Parallel Repetition of the Odd Cycle Game

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Abstract. Higher powers of the Odd Cycle Game has been the focus of recent investigations [3,4]. In [4] it was shown that if we repeat the game d times in parallel, the winning probability is upper bounded by $1 - \Omega(\frac{\sqrt{d}}{n\sqrt{\log d}})$, when $d \leq n^2 \log n$. We

1. Determine the exact value of the square of the game;
2. Show that if Alice and Bob are allowed to communicate a few bits they have a strategy with greatly increased winning probability;
3. Develop a new metric conjectured to give the precise value of the game up-to second order precision in $1/n$ for constant d.
4. Show an $1 - \Omega(d/n \log n)$ upper bound for $d \leq n \log n$ if one player uses a "symmetric" strategy.

Keywords: parallel repetition, two prover games, CHSH

1 Introduction

It is well known due to the famous parallel repetition theorem of Ran Raz, that higher powers of any two-prover game have exponentially decreasing values:

Theorem 1 (Parallel Repetition Theorem [10]). *For every fixed answer size c and for every $\epsilon > 0$ there is a $\delta > 0$ such that if $v(G) < 1 - \epsilon$ then $v(G^d) < (1 - \delta)^d$ for every $d \geq 1$. Moreover, for fixed c we have $\delta \in \Omega(\epsilon^{32})$.*

A simplification by Hollenstein improves on the dependence between ϵ and δ:

Theorem 2 (Hollenstein [7]). *In Theorem 1 $\delta \in \Omega(\epsilon^3)$ for fixed c.*

We will focus on the Odd Cycle Game, which is a special XOR game. XOR games are two-prover games for which the answers, a of Alice and b of Bob are binary, and to every question pair there is a $rel \in \{=, \neq\}$ such that the answers are good iff a rel b. For the n-cycle game, G_n (n is odd), $v(G_n) = 1 - 1/2n$. Since this value tends to 1, when $n \to \infty$, the game is a candidate for a counter-example to:

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Conjecture 1 (Strong parallel repetition conjecture [4]). $\delta \in \Omega(\epsilon)$.

The conjecture, among others, would imply that to prove the famous Unique Game Conjecture it would be sufficient to prove the NP-hardness of the gap problem $MAXLIN2(1-\epsilon, 1-\sqrt{\epsilon})$ [4]. Feige and Lovász [5], and later Cleve et.al. proved that $\delta \in \Omega(\epsilon^2)$ for XOR games [3]. Both proofs are based on the duality theorem for semidefinite programming and a clever product theorem.

Uri Feige, Guy Kindler and Ryan O'Donnell have recently shown that $v(G_n^d) = 1 - \Omega(\frac{\sqrt{d}}{n\sqrt{\log d}})$ (for $d \leq n^2 \log n$) by a novel geometric intuition [4], thus improving on [5,3] for the odd cycle games for $d < n^{2-c}$ ($c > 0$). They also showed that improving their bound requires improving lower bounds on the surface area of high-dimensional foams. In other words if someone wants to improve on the upper bound for the odd cycle games (in particular, if one wants to prove the Strong parallel repetition conjecture), she also needs to improve on the best lower bound to the following hard question: What is the least surface area of a cell that tiles R^d by the lattice Z^d? We need to note that the version of odd cycle game they discussed is sightly (but not essentially) different from our version. We list some further differences compared to [4]:

1. We have found tight bounds for the two rounds repetition of our version.
2. In [4] the connection between geometry and the odd cycle games is one-directional, while in our case, for $d = 2$ it is two-directional.
3. In Section 3, we give a meaning to the "2-cycle game," which turns out to be identical to the so-called CHSH game (Section 4). The connection might provide a hint to the exact general formula for $v(G_n)^d$, when d is small: Strategies for small powers of the CHSH game can be searched with computer. Conversely, for powers of the CHSH game the geometrical approach presented in this paper and in [4] may turn out to be the right approach.

We develop a topological machinery for our version of the odd cycle game and invent a new, interesting metric. Our approach, although does not represent an essential departure from [4], may provide additional insight. In particular, we believe, the connection between the topology and the game is more transparent in our discussion. More importantly, due to our precise metric, we are potentially able to obtain the exact constants for small powers of the game up to the second order term in n.

In the second half of the paper we extend our investigation to the case when the two players can communicate, and achieve winning probability 1 or close to

1 for linear number of repeats. We also show that if either Alice or Bob plays a *symmetric* strategy, then the value of the strategy almost meets the strong parallel repetition bound.

2 A Syntactic Aside

To avoid the nightmare of "onion-ized" expressions when dealing with iterated moduli of the type ((expr (mod a)) (mod b)), we introduce the following convention: $\tilde{+}$ is the operator that adds two integers mod n and returns an integer in $\{-\lfloor\frac{n-1}{2}\rfloor, \ldots, \lceil\frac{n-1}{2}\rceil\}$; $\oplus$ is the mod 2 addition. It takes two integers and returns their sum mod 2. The result is an integer in $\{0, 1\}$. If the left hand side of an equation is mod addition, the reader needs to reduce the the right hand side with the same modulus. Sometimes we need to forcefully reduce both sides of an equation by the mod 2 operator. We extend all operators to vectors, coordinate-wise. The $=$, $\geq$ and other relations hold to vectors if they hold for all coordinates.

3 The Odd Cycle Game and Its Powers

Let n be an odd number. The *n-cycle game*, G_n, starts with Alice and Bob picking a pair of colorings of the n-cycle, $\mathcal{S}_A, \mathcal{S}_B : [n] \to \{0, 1\}$, called strategies. The verifier then selects $0 \leq x < n$ and a type $t \in \{0, 1\}$, both randomly and accepts iff $\mathcal{S}_A(x) \oplus \mathcal{S}_B(x \tilde{+} t) = t$. The best strategy $\mathcal{T} = (\mathcal{S}_A, \mathcal{S}_B)$ is where $\mathcal{S}_A(x) = \mathcal{S}_B(x) = x \mod 2$. $v(G_n) = v(\mathcal{T}) = 1 - 1/2n$.

Remark 1. In [4] $t \in \{0, 1, -1\}$ randomly, and the test is the same.

The verifier of the dth power of the n-cycle game selects a tuple $\boldsymbol{x}$ from $[n]^d$ randomly and a type $\boldsymbol{t}$ from $[2]^d$ randomly. The strategies of Alice and Bob are: $\mathcal{S}_A, \mathcal{S}_B : [n]^d \to [2]^d$. The verifier then evaluates the following predicate:

$$\mathcal{S}_A(\boldsymbol{x}) \oplus \mathcal{S}_B(\boldsymbol{x} \tilde{+} \boldsymbol{t}) \stackrel{?}{=} \boldsymbol{t}. \qquad (1)$$

By definition, the equality of the two sides means that the acceptance criterion has to hold for all coordinates.

4 Even Cycle Games

The so-called CHSH game has received considerable attention in quantum physics due to the Einstein, Podolsky, Rosen paradox. To single bit questions x and y

the single bit answers $\mathcal{S}_A(x)$, $\mathcal{S}_B(y)$ are accepted, iff

$$\mathcal{S}_A(x) \oplus \mathcal{S}_B(y) = xy.$$

The value of this game is 0.75. The paradox arises in the quantum world, where Alice and Bob can win the game with probability 0.85 by communicating through a pair of entangled electrons. The communication is instant even when Alice and Bob are light years apart, and the evidence that communication has happened is the increased value of the game. This met Einstein's disapproval. Let us generalize the acceptance criterion of the n-cycle game, (n is odd) as:

$$\mathcal{S}_A(x) \oplus \mathcal{S}_B(x \tilde{+} t) \stackrel{?}{=} \ t \oplus \delta(x,t),$$

where $\delta : [n] \times \{0,1\} \to \{0,1\}$ is any function with $\sum_{n,t} \delta(n,t) = 0 \mod 2$. The transformation preserves all properties of the game, including its and its powers' values. To extend the notion to even ns we just replace the condition on δ with

$$\sum_{n,t} \delta(n,t) = n + 1 \mod 2. \tag{2}$$

Regardless of the parity of n, the value of the n cycle game is $1 - \frac{1}{2n}$. Although for simplicity we discuss only odd cycle games, all our arguments carry over to even cycle games. The CHSH game arises from $\delta(x,t) = xt - x - t \mod 2$ (one can check that (2) holds). The value of CHSH^2 is 5/8 (this also follows from our results). By exhaustively searching all strategies, Aaronson and Toner [3] independently have found that $v(\text{CHSH}^3) = 31/64$, while any strategy, where the first two rounds are independent of the third have value at most $5/8 * 3/4 = 30/64$. This shows that something interesting is happening for the third power too.

5 Local Consistency and Pearls

For fixed $\mathcal{S}_A$ let Bob optimize over his strategies: $v(\mathcal{S}_A) = \max_{\mathcal{S}_B} v(\mathcal{S}_A, \mathcal{S}_B)$. Bob now has to optimize only locally: For every $\boldsymbol{y} \in [n]^d$ Bob needs to set $\mathcal{S}_B(\boldsymbol{y})$ such that the number of $\boldsymbol{t}$s for which $\mathcal{S}_A(\boldsymbol{y} \tilde{-} \boldsymbol{t}) \oplus \mathcal{S}_B(\boldsymbol{y}) = \boldsymbol{t}$ is maximized. Putting it differently, Bob needs to pick the plurality value of $\mathcal{S}_A(\boldsymbol{y} \tilde{-} \boldsymbol{t}) \oplus \boldsymbol{t}$ on the cube $Q_{\boldsymbol{y}} = \{\boldsymbol{y} \tilde{-} \boldsymbol{t} \mid \boldsymbol{t} \in \{0,1\}^d\}$. We say that $\boldsymbol{x}, \boldsymbol{x'} \in Q_{\boldsymbol{y}}$ are consistent iff there exists an answer B for Bob to $\boldsymbol{y}$, which is consistent with both $\mathcal{S}_A(\boldsymbol{x})$ and $\mathcal{S}_A(\boldsymbol{x'})$. This is the case if and only if: $\mathcal{S}_A(\boldsymbol{x}) - \mathcal{S}_A(\boldsymbol{x'}) = \boldsymbol{x} \tilde{-} \boldsymbol{x'} \mod 2$. *Notice that consistency is independent of* $\boldsymbol{y}$*!!*

Definition 1 (consistency of a region). *A region $R \subseteq [n]^d$ is* consistent *(w.r.t. $\mathcal{S}_A$) if for every $\boldsymbol{x}, \boldsymbol{x'} \in R$ it holds that $\mathcal{S}_A(\boldsymbol{x}) - \mathcal{S}_A(\boldsymbol{x'}) = \boldsymbol{x} \tilde{-} \boldsymbol{x'} \mod 2$.*

Whether or not two points of $[n]^d$ are consistent, from the point of view of computing $v(\mathcal{S}_A)$ is interesting only locally. Define $R_{\boldsymbol{y}}$ as the maximal consistent sub-region of $Q_{\boldsymbol{y}}$ (if there are more than one, break the tie arbitrarily). Then:

$$v(\mathcal{S}_A) = \frac{1}{n^d} \sum_{\boldsymbol{y} \in [n]^d} \frac{|R_{\boldsymbol{y}}|}{2^d}. \tag{3}$$

What prevents $R_{\boldsymbol{y}} = Q_{\boldsymbol{y}}$ for all $\boldsymbol{y} \in [n]^d$? We give a completely topological explanation. We call a sequence $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k \in [n]^d$ *cycle* if $\boldsymbol{x}_k = \boldsymbol{x}_0$.

Definition 2. *Let n be odd. A cycle $C = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_k \in [n]^d$; $\boldsymbol{x}_k = \boldsymbol{x}_0$ is*

$$\textbf{topologically non-trivial} \quad \textit{iff} \quad \sum_{i=0}^{k-1} \boldsymbol{x}_{i+1} \tilde{-} \boldsymbol{x}_i \neq \boldsymbol{0}.$$

$$\textbf{topologically odd} \quad \textit{iff} \quad \sum_{i=0}^{k-1} \boldsymbol{x}_{i+1} \tilde{-} \boldsymbol{x}_i \neq \boldsymbol{0} \mod 2$$

Definition 3 (Pearl, Consistent Pearl). *A* pearl *$\wp$ is a collection $\{R_{\boldsymbol{y}} | \boldsymbol{y} \in [n]^d\}$ such that $R_{\boldsymbol{y}} \subseteq Q_{\boldsymbol{y}}$ for all $\boldsymbol{y} \in [n]^d$. A pearl $\wp$ is* consistent *(with respect to $\mathcal{S}_A$) if all regions $R_{\boldsymbol{y}}$ are consistent (with respect to $\mathcal{S}_A$).*

The value of the n-cycle game is the maximum of $\frac{1}{n^d 2^d} \sum_{\boldsymbol{y} \in [n]^d} |R_{\boldsymbol{y}}|$, where $\{R_{\boldsymbol{y}}\}_{\boldsymbol{y}}$ is *some* consistent pearl for *some* $\mathcal{S}_A$. To translate the maximization problem to a purely combinatorial problem we characterize these pearls.

Definition 4. *A cycle $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k \in [n]^d$; $\boldsymbol{x}_0 = \boldsymbol{x}_k$ is* contained *in $\wp = \{R_{\boldsymbol{y}}\}_{\boldsymbol{y}}$, if there are $R_0, \ldots, R_{k-1} \in \wp$ such that $\boldsymbol{x}_i, \boldsymbol{x}_{i+1} \in R_i$ for $0 \leq i \leq k-1$.*

Lemma 1. *For fixed $\mathcal{S}_A$ let $C = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_k$ ($\boldsymbol{x}_k = \boldsymbol{x}_0$) be such that $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ are consistent w.r.t. $\mathcal{S}_A$ for $0 \leq i \leq k-1$. Then C is a topologically even cycle.*

Lemma 2 (Main Lemma). *$\wp = \{R_{\boldsymbol{y}}\}_{\boldsymbol{y}}$ is a consistent pearl with respect to some $\mathcal{S}_A$, or shortly a* consistent pearl*, if and only if it does not contain a topologically odd cycle.*

6 The Topological Approach

Let $T_d = (0,1]^n \subseteq \mathbf{R}^n$ be the d dimensional unit torus and $n \times T_d = (0,n]^n$. Consider a cycle $C = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_k$ ($\boldsymbol{x}_k = \boldsymbol{x}_0$) in $[n]^d$ (n is odd) . We can naturally embed $[n]^d$ into $n \times T_d$. If we connect each $\boldsymbol{x}_i$ with $\boldsymbol{x}_{i+1}$ via the geodesics Γ_i in $n \times T_d$, we get a closed curve, $\Gamma = \Gamma(C) = \cup_i \Gamma_i$. We can then study the group element $g(\Gamma)$ of the homotopy group $\pi_1(T_d)$, associated with Γ. It is well known that $\pi_1(T_d) = \mathbf{Z}^d$, where the i^{th} coordinate of $g \in \pi_1(T_d)$ tells how many times a curve wraps around the cycle in the i^{th} coordinate direction. The following lemma, which justifies the terms "topologically trivial" and "odd," is easy to prove:

Lemma 3. *Let C be a cycle in $[n]^d$ (n is odd). Then C is topologically trivial (even) if and only if $g(\Gamma(C)) = \mathbf{0}$ ($g(\Gamma(C)) \in (2\mathbf{Z})^d$).*

Corollary 1. *A pearl $\wp$ is consistent if and only if whenever C is a cycle of $\wp$, $g(\Gamma(C)) \in (2\mathbf{Z})^d$.*

7 Blockers

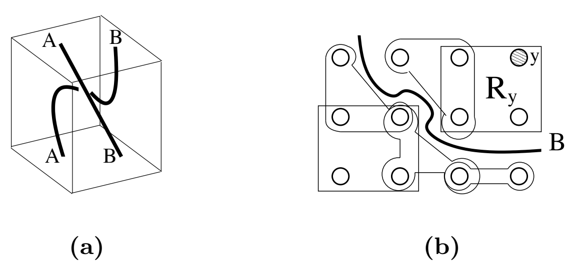


Fig. 1. (a) Even, but non-trivial cycle **(b)** Part of a pearl created by blocker B

Blockers are subsets of T_d that intersect with all cycles that are not in the homotopy class of $\mathbf{0}$. Blockers are called *foams* in [4]. *Odd blockers* are subsets of T_d that intersect with all cycles whose homotopy class is not in $(2\mathbf{Z})^d$. We can construct blockers and odd blockers from $d-1$ skeletons of cell complexes. For an intuition the reader can skip to Section 9, that discusses the case of $d = 2$.

$\wp_n(B)$ (Odd blockers $\rightarrow$ consistent pearls): Let B be an odd blocker of T_d such that $(n \times B) \cap [n]^d = \emptyset$. For any $\boldsymbol{y} \in n \times T_d$ we define the solid cube $Q^*_{\boldsymbol{y}} = \boldsymbol{y} \hat{-} Q^*$, where Q^* is $[0,1)^n$ and $\hat{-}$ is the wrap-around subtraction inside $n \times T_d$. We then define an equivalence relation between the elements of $Q_{\boldsymbol{y}}$: $\boldsymbol{x}, \boldsymbol{x'} \in Q_{\boldsymbol{y}}$ are equivalent if they can be connected inside $Q^*_{\boldsymbol{y}}$ without intersecting $n \times B$. Let

$R_{\boldsymbol{y}}(B)$ be a maximum size equivalence class (we break ties in some arbitrary systematic manner). We define the pearl $\wp_n(B) = \{R_{\boldsymbol{y}}(B)\}_{\boldsymbol{y}}$.

Lemma 4. *Let $n \geq 3$, odd, B be an odd blocker in T_d, Then pearl $\wp_n(B)$ is consistent.*

Proof. Let $C = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_k$ ($\boldsymbol{x}_k = \boldsymbol{x}_0$) be a cycle in $\wp_n(B)$. We need to prove that C is not topologically odd. Since C is *in* $\wp_n(B)$, there exist $\boldsymbol{y}_i \in [n]^d$ such that $\boldsymbol{x}_i, \boldsymbol{x}_{i+1} \in Q_{\boldsymbol{y}_i}$ (Definition 4). By the definition of $\wp_n(B)$ we can construct a curve Γ_i' that connects $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$, runs inside $Q^*_{\boldsymbol{y}_i}$, and which is not blocked by B. Since $\Gamma' = \cup_i \Gamma_i'$ is not blocked by B, we have that $g(\Gamma') \in (2\mathbf{Z})^d$. For Corollary 1, however, we need $g(\Gamma(C)) \in (2\mathbf{Z})^d$.

Definition 5. *For curves Γ and Γ' define $|\Gamma, \Gamma'|_\infty$ as $\inf_\phi |x, \phi(x)|_\infty$, where ϕ is a one-one continuous map between Γ and Γ'.*

Since $|\Gamma', \Gamma(C)|_\infty \leq 1$, we can apply the following lemma:

Lemma 5. *if Γ, Γ' are curves in $n \times T_d$ and $|\Gamma, \Gamma'|_\infty < \frac{n}{2}$, then $g(\Gamma) = g(\Gamma')$.*

8 A New Metric

Let S be a piece of a smooth $d-1$ dimensional surface (or a union of these) in T_d such that $(n \times S) \cap [n]^d = \emptyset$ for every $n \geq 1$. Create the pearl $\wp_n(S) = \{R_{\boldsymbol{y}}\}_{\boldsymbol{y}}$ in the same way as in Section 7 when S was an odd blocker. Although now $\wp_n(S)$ is not (necessarily) consistent, we can still associate the value $v_n(S) = \frac{1}{2^d n^d} \sum_{\boldsymbol{y} \in [n]^d} |R_{\boldsymbol{y}}|$ to it. It turns out that the measure

$$\lim_{n \to \infty} n(1 - v_n(S)) = \lambda(S)$$

exists, and it is additive in the sense that if S is a disjoint union of pieces $S_1, \ldots, S_m$ then $\lambda(S) = \sum_{i=1}^m \lambda(S_i)$. What is this measure? If S is a piece of a $d-1$ dimensional hyper-plane with normal vector $S = (s_1, \ldots, s_d)$ (where s_i is the projection size of S on the i^{th} coordinate plane), then

$$\lambda(S) = \frac{1}{2} E\left(|\sum_{i=1}^d s_i \chi_i| \right), \tag{4}$$

where χ_i are independent $\{1, -1\}$-valued uniform random variables.

Definition 6 (Diamond norm). *For vector $A = (a_1, \ldots, a_n)$ define its diamond norm as $|A|_\diamond = E(|\sum_{i=1}^d a_i \chi_i|)$, where χ_i are independent $\{1, -1\}$-valued uniform random variables.*

Lemma 6. *$|A|_\Diamond = |A|_\infty$, when $d = 2$, and $|A|_\Diamond \geq |A|_\infty$ otherwise.* $\frac{|A|_2}{\sqrt{2}} \leq |A|_\Diamond \leq |A|_2$.

Only the $\frac{|A|_2}{\sqrt{2}} \leq |A|_\Diamond$ relation is hard to show, which comes from the Khintchine inequality. Notice that the diamond norm is the same as the L_2 norm within a constant factor, which explains [4]. The additivity of λ and (4) gives:

Theorem 3. *Let odd blocker B be the $d-1$-skeleton, of a smooth cell complex. Assume that $(\forall n > 0)$ $(n \times B) \cap [n]^d = \emptyset$. Then for every $\epsilon > 0$ there exists an n_ϵ such that the strategy associated with $\wp_n(B)$ $(n \geq n_\epsilon)$ has value at least*

$$1 - \frac{1+\epsilon}{2n} \iint_B |dB|_\Diamond.$$

It is unclear to us if there is any strategy with greater value than the one that arises from the best odd blocker or even from the best blocker. In [4] it is conjectured that blockers give the best strategies. They also show that $v(G_n^d) = 1 - \Omega(\frac{\sqrt{d}}{n\sqrt{\log d}})$ for $d \leq n^2 \log n$.

9 The Case of $d = 2$

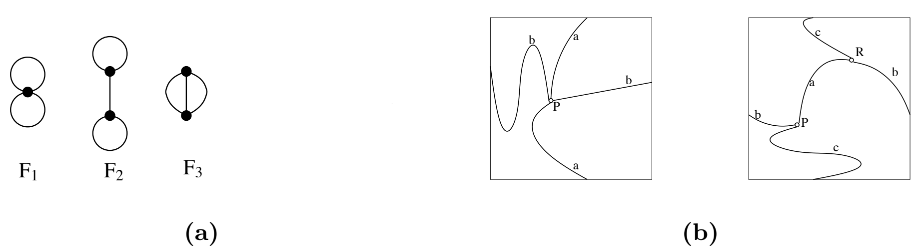


Fig. 2. **(a)** shows the three combinatorially possible graphs **(b)** shows the two of the three that are torical

Two dimensional cell complexes on the torus are simply graphs drawn on the torus. A graph drawn on the torus is torical (i.e. "takes the full use of the torus") if its edges block all non-trivial cycles.

Theorem 4. *Every topologically non-trivial* simple cycle *in the two dimensional torus is also topologically odd.*

Corollary 2. *A graph on T_2 is torical iff its edges block all the odd cycles.*

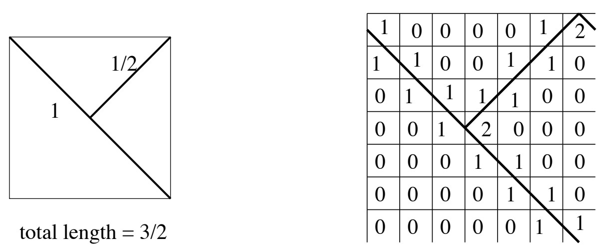
If a torical graph creates more than one facets we can delete at least one of its edges and remain torical. For a torical graph the Euler's theorem gives:

$$-f + e - v = 0,$$

where f is the number of faces, e is the number of edges and v is the number of vertices. We can assume that all vertices have degree at least 3. This gives $e \geq \frac{3}{2}v$. Thus if $f = 1$, the possible parameter combinations are $v = 1, e = 2$ and $v = 2, e = 3$. This gives us three graphs, F_1, F_2, and F_3 (see Figure 2). Only F_1 and F_3 have torical representations. Furthermore, F_1 can be viewed as a special case of F_3, where one of the edges is shrunk to a point.

Lemma 7. *The minimum total edge-length of any torical representation of F_3 on the unit torus is at least 1.5. Above all lengths are measured in the L_∞ norm (hence in the Diamond norm).*

Proof. As in Figure 2 **b** we denote the two nodes of F_3 by P and R. Let the length of the shorter horizontal projection of $\overline{PR}$ be $x \leq 0.5$ and the length of the shorter vertical projection be $y \leq 0.5$. Without loss of generality we can assume that the L_∞ length of one of the three edges is at most 0.5. This edge has lengths at least $\max\{x, y\}$ and then the other two have length at least $\max\{x, 1 - y\}$ and $\max\{1 - x, y\}$, respectively. Assume $x \geq y$. For the total L_∞ length of the graph we now get $x + (1 - x) + (1 - y) \geq 1.5$.



(a) (b)

Fig. 3. (a) An optimal blocker in the two-dimensional unit torus with respect to the L_∞ norm. **(b)** An optimal strategy for $n = 7$ arising from the blocker on the left. The torus is scaled up by a factor of n. Losses are shown in each square.

Figure 3 **a** shows that 1.5, is achievable and Figure 3 **b** shows how this gives rise to a strategy $\mathcal{S}_2$ with value exactly $1 - \frac{3}{4n}$. The number in each square denotes the "loss" $4-|R_{\boldsymbol{y}}|$. One can easily see that the total loss is precisely $3n$.

Theorem 5. $v(\mathcal{S}_2) = 1 - \frac{3}{4n}$.

We also give a matching upper bound relying on Lemma 7.

Theorem 6. $v(G_n^2) = 1 - \frac{3}{4n}$.

Proof. We need to prove the upper bound. Consider a strategy $\mathcal{S}_A$ of Alice and associate a torical graph, G, with it. We define "portions" of G in each square $Q^*_{\boldsymbol{y}}$, using the consistency classes created by $\mathcal{S}_A$. Instead of a detailed explanation we refer the reader to Figure 4. In each Q^*_x the total L_∞ length of the portion of G is $\frac{1}{2}(4-|R_x|)$ thus $\sum_{e\in E(G)} |e|_\infty = 2n^2(1-v(\mathcal{S}_A))$ by (3) ($|e|_\infty$ is the L_∞ length of e).

Consider an arbitrary *simple* (i.e. not self-intersecting) cycle Γ' in T_2 that does not intersect G. We show that $g(\Gamma') = \mathbf{0}$. Pick an orientation for Γ', and a staring point P in it. We define a cycle $C = \boldsymbol{x}_0, \ldots, \boldsymbol{x}_k$ ($\boldsymbol{x}_k = \boldsymbol{x}_0$) by the following algorithm. We start with the empty sequence, and walk along Γ' from P. Whenever we leave the current $Q^*_{\boldsymbol{y}}$, we look for the grid point that is closest (in L_∞ norm) to the point we exit $Q^*_{\boldsymbol{y}}$, and we add this grid point to the sequence. We continue until we get back to P, and pass a little further until we add $\boldsymbol{x}_k$, which is $\boldsymbol{x}_0$. The main observation is that $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ are consistent for $0 \le i \le k-1$. This comes from reviewing Figure 4 (a). By Lemma 1 this implies that C is topologically even. Now $|\Gamma', \Gamma(C)| < 1$ together with Lemmas 3, 5 and Corollary 2 imply that G is torical, which in turn, by Lemma 7 gives that $\sum_{e\in E(G)} |e|_\infty \ge 1.5n$. Hence $1.5n \le 2n^2(1-v(\mathcal{S}_A))$, as needed.

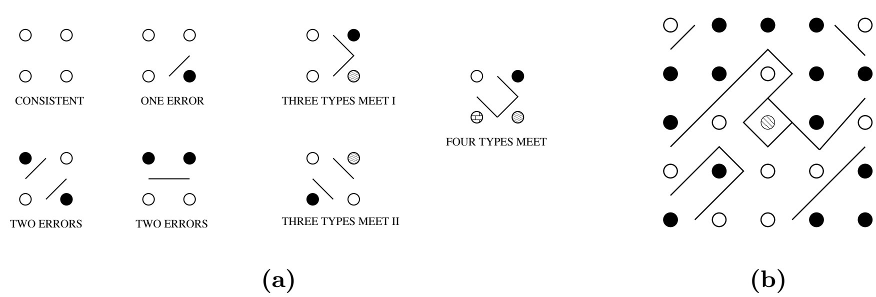


Fig. 4. **(a)** Turning a strategy into lines **(b)** Part of the emerging graph

10 Gap Commitment Problem

The starting point of this research was the following question of Gavinsky: Is it true that $v(G_n^d) = 1 - \Omega(1)$ for $d = n$? When $d \approx n$, non-topological type strategies could be promising. In the rest of the article we discuss such a "different" type of strategy.

Let us think of the question vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ to Alice and Bob as d-element multi-sets of Z_n. Let $\underline{\boldsymbol{x}} = \{x_i\}_{i=1}^d$, $\underline{\boldsymbol{y}} = \{y_i\}_{i=1}^d$ be their supporting sets. When $d = n$, there are typically points and short intervals missing from $\underline{\boldsymbol{x}}$ (and from $\underline{\boldsymbol{y}}$). We call such a interval *gap*. Since the provers receive different questions, they see different gaps in their multi-set, but since their questions correlate, so will the gaps. We say that $\alpha \in Z_n$ is in a gap of size l for a verifier's question $\underline{\boldsymbol{x}}$ to Alice, if the interval $\{\alpha, \alpha \pm 1, \ldots, \alpha \pm l\}$ is disjoint from $\underline{\boldsymbol{x}}$.

The main idea is that if Alice and Bob can agree with probability $1 - \epsilon$ over the verifier's question pair $(\boldsymbol{x}, \boldsymbol{y})$ in some $\alpha \in Z_n$ such that α is in some gap with respect to both $\boldsymbol{x}$ and $\boldsymbol{y}$, then they can win the game with probability $1 - \epsilon$. Indeed, to every x_i Alice can answer with $x_i \hat{-} \alpha \mod 2$ (Bob with $y_i \hat{-} \alpha \mod 2$), where $\hat{-}$ returns the mod n value of the difference in the non-negative representation. A refinement of this idea is that it is sufficient if Alice and Bob find gaps with respect to their inputs with non-empty intersection. Such agreement actually seems easy at first: both players just pick their largest gap. The catch is that their largest gaps will be totally different with constant probability. In the above algorithm Alice (Bob too) plays a *symmetric* strategy: Her answers depend only on the mlti-set of $\boldsymbol{x}$. For symmetric strategies we have found a bottleneck:

Theorem 7. *Assume that Alice plays a symmetric strategy and $d = \Omega(n \log n)$. Then regardless of Bob's strategy the winning probability is at most $1 - \Omega(1)$.*

The above strategy works, however, if we allow a small amount of communication. **Gap Commitment Problem (GCP):** The input to both Alice and Bob are multi-sets from Z_n of size d. How many bits of communication is required to find an x, which is in a gap for both Alice and Bob? We denote this problem by GCP_n^d. It seems that GCP is an interesting problem on its own right.

Let D denote the deterministic one-round communication complexity and let D_ϵ denote the one round (Distributional) communication complexity (by a deterministic protocol) which is allowed to fail on ϵ fraction of all inputs (see [8]), where the distribution of $(\boldsymbol{x}, \boldsymbol{y})$ is the same as in the odd cycle game.

Theorem 8. *Let $d < n/10$, $t \geq 1$. Then:*

$$D(GPC_n^d) \in O(\log n),$$
$$D_{\frac{1}{t}}(GPC_n^d) \in O(\log \log t).$$

Proof. In the first case, Alice can send the location of a point in a gap with size at least 2 to Bob. Since $d < n/10$, such a point exists. This point is in a gap with size at least 1 for Bob.

In the second case Alice looks at $\{1, \ldots, \lceil 5 \log t \rceil\}$ and selects a gap in this set with size of at least 2 with center g, if exists. She just needs $O(\log \log t)$ bits (one-round) communication to report the location of g to Bob.

Remark 2. Although the above strategies are simple, they demonstrate that no strong direct sum theorems hold for the communication version of the odd cycle game: $D(G_n) = 2 \gg \frac{10}{n} D(G_n^{n/10}) \in O(\frac{\log n}{n})$. Different amortized measures were studied by I. Parnafes, R. Raz, and A. Wigderson [9], and also, by T. Feder, E. Kushilevitz, M. Naor, and N. Nisan [6] and by Ambainis et. al. [1].

References

1. Ambainis, A., Buhrman, H., Gasarch, W., Kalyanasundaram, B., Torenvliet, L.: The Communication Complexity of Enumeration, Elimination, and Selection. Journal of Computer and System Science 63(2), 148-185 (2001).
2. Clauser, J., Horne, M. A., Shimony, A., Holt, R. A: . Phys. Rev. Lett. 23, 880 (1969).
3. Cleve, R., Slofstra, W., Unger, F., Upadhyay, S.: Perfect Parallel Repetition Theorem for Quantum XOR Proof Systems. IEEE Conference on Computational Complexity 2007: 109-114
4. Feige, U., Kindler, G., O'Donnell, R.: Understanding parallel repetition requires understanding foams. IEEE Conference on Computational Complexity 2007: 179-192.
5. Feige, U., Lovász: Two-prover one-round proof systems: their power and their problems. Proc. 24th ACM Symp. on Theory of Computing (1992), 733–744.
6. Feder, T., Kushilevitz, E., Naor, M.,Nisan, N.: Amortized communication complexity. SIAM Journal of Computing, 239-248, (1995).
7. Holenstein, T.: Parallel repetition: simplifications and the no-signaling case. to appear in Proc. of 39th STOC, (2007).
8. Kushilevitz, E., Nisan, N.: Communication Complexity. Cambridge University Press, (1997).
9. Parnafes, I., Raz, I., Wigderson, A.: Direct Product Results and the GCD Problem, in Old and New Communication Models. Proc. of the 29th STOC, 363-372 (1997).
10. Raz, R.: A Parallel Repetition Theorem. Siam Journal of Computing 27(3), 763-803 (1998).