

1. **The Random Walk on the Integers:** Start with the **coin toss game** which uses a coin with probability \mathcal{P} for Head and probability $q = 1 - \mathcal{P}$ for Tail. You get a dollar if it comes up Head and lose a dollar if it shows Tail. The coin will be tossed infinitely many times and the random variable X_i measures your earnings on the i^{th} toss:

$$X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ toss is Head} \\ -1 & \text{otherwise} \end{cases}$$

Your profit (excess of heads over tails) after m tosses is

$$S_m = X_1 + \cdots + X_m.$$

Your progress in this game may be thought of as describing **a random walk on the integers**. Let $S_0 = 0$ signify the start at the origin. Each toss is a *step* in which “Head” moves you one integer to the right from the current position and “Tail” moves one integer to the left. Thus S_j is the position of the walk after the j^{th} step (toss). The study of this interesting game reveals some surprising properties, and it is another very significant example showing the power of generating functions.

- **Positive Paths - The Ballot Theorem** Another way to describe this walk uses the sequence of points $\{(0, 0), (1, S_1), (2, S_2), \dots, (m, S_m)\}$; after step j our point has x -coordinate j and y -coordinate $S_j = X_1 + \cdots + X_j$, the *excess* of heads over tails. If we connect successive points by straight line segments we get a **random walk path** with m steps (starting at the origin, ending at (m, S_m)). There are 2^m such paths (because each sequence of m tosses gives a distinct path). We will take $\mathcal{P} = 1/2$, so each of them is equally likely.

We write

$$m = n_H + n_T$$

for the number of tosses, n_H denoting the number of Heads and n_T the number of Tails, and observe that the score after m tosses is the difference between Head and Tail, or

$$S_m = n_H - n_T.$$

We fix a positive integer k , $0 < k \leq m$, and focus our attention on the paths that go from $(0, 0)$ to (m, k) ; i.e. $S_m = k$ is the score after m steps. Since $S_m = n_H - n_T = 2n_H - m$, k is even if m is, and vice-versa. Also note that the number of distinct paths which start at $(0, 0)$ and end at (m, k) is

$$N_{m,k} = \binom{m}{n_H},$$

each one equally likely. A path is called **GOOD** (positive) if $S_j > 0$, for all j , $1 \leq j \leq m$; i.e., it is positive after the start. The ballot theorem says that the number of good paths from $(0, 0)$ to (m, k) is

$$G_{m,k} = N_{m,k} \left(\frac{n_H - n_T}{m} \right) = \frac{k}{m} \binom{m}{n_H} \quad (1)$$

This gives k/m as the probability that a path from $(0, 0)$ to (m, k) is good. The statement in (1) is called the Ballot Theorem because if Candidate A gets n_H votes and Candidate B gets $n_T < n_H$ votes (so there are $m = n_H + n_T$ ballots and A wins by $k = n_H - n_T$ votes), and the m ballots are counted in a random order, each of the $m!$ orderings being equally likely, the probability k/m describes the chances that A leads throughout the counting of the ballots [its the fraction of the $m!$ orderings where A is always in the lead. It is interesting to note that this probability is the slope of the line joining $(0, 0)$ to (m, k) . Note that if candidate A wins by only a small margin, the probability that he was always ahead during the counting is small (think of the recent presidential election in Florida).

We apply the Ballot Theorem to the paths that start at $(-1, -1)$ and end at (m, k) , where now we take $k \geq 0$ (i.e., zero is allowed). We call a path “positive” if, after the start, it is strictly above $y = -1$. Every such path (once you reach $(0, 0)$) is a true nonnegative path from $(0, 0)$ to (m, k) . These so-called “positive” paths from $(-1, -1)$ have $n_H + 1$ Heads and n_T Tails, so (1) gives

$$N_{m,k}^* = \frac{k+1}{m+1} \binom{m+1}{n_H+1} \quad (2)$$

as the number of **non-negative paths** from $(0, 0)$ to (m, k) . Dividing by $N_{m,k}$ reveals that the probability of a non-negative path from $(0, 0)$ to (m, k) is

$$\frac{k+1}{n_H+1}.$$

An interesting special case of (2) is when $n_H = n_T = n$, so $k = 0$. and $m = 2n$. This gives

$$N_{2n,0}^* = \frac{1}{2n+1} \binom{2n+1}{n+1} = \frac{1}{n+1} \binom{2n}{n},$$

the last expression easily deduced from the middle one. This shows that the set of n node binary trees and the set of non-negative random walk paths from $(0, 0)$ to $(2n, 0)$ have **the same size!!**

- **First Lead of \$1:** Write \mathcal{P} for the probability of Head, not assumed to be $1/2$. We study the probability of the event A_n that the walk *first* arrives at the integer 1 at the n^{th} step (you lead by 1 after step n but were even, or behind, previously). Thus

$$A_n = \{S_n = 1, \text{ and } S_j < 1, j < n\}.$$

Let X be a random variable (on the sample space of coin tosses) that counts the number of trials you need to first achieve a lead of 1 over the opponent (who gets \$1 for Tail and loses \$1 for Head). A_n is the event $\{X = n\}$ and we write

$$p_n = f_X(n) = \text{Prob}(X = n)$$

for its probability. Clearly $p_1 = \mathcal{P}$ (Head on first toss) and $p_2 = 0$ (HH, HT, TH, TT gives $S_2 = 2, 0, 0, -2$, respectively. In fact $p_{2j} = 0$ since it is impossible to lead by 1

after an even number of trials). Also $p_3 = (1 - \mathcal{P})\mathcal{P}^2$ and $p_5 = 2(1 - \mathcal{P})^2\mathcal{P}^3$ (THTHH or TTHHH). We will compute the generating function of X ,

$$\phi_X(s) = \sum_{k=1}^{\infty} p_k s^k.$$

Take $n > 1$ and note that $\{X = n\}$ occurs only if for some $k, 2 \leq k < n$

- (a) the first trial fails (this is phase I)
- (b) you first equalize Heads and Tails at trial k (phase II comprises trials $2, \dots, k$)
- (c) you take $n - k$ additional trials to first get a lead of 1 (phase III covers trials $k + 1, \dots, n$)

Clearly (a) is necessary for A_n , and if you are trailing by 1 at time 1 and will be leading by 1 at time n , you must be equal sometime; k is when that first occurs.

Let X_1 measure the number of trials (after trial 1) for phase II and X_2 the number of trials for phase III. Letting C be “tail on trial 1”, we have

$$\{X = n\} = \bigcup_{k=2}^{n-1} (C \cap \{X_1 = k - 1\} \cap \{X_2 = n - k\})$$

and because the events in the union are mutually exclusive,

$$p_n = \sum_{k=2}^{n-1} \text{Prob}(C \cap \{X_1 = k - 1\} \cap \{X_2 = n - k\}).$$

The three events in the intersections are independent because the phases are non-overlapping Bernoulli trials, and since the X_i 's, like X , measure the number of trials needed to first get a lead of 1,

$$p_n = q \sum_{k=2}^{n-1} p_{k-1} p_{n-k} = q(p_1 p_{n-2} + \dots + p_{n-2} p_1)$$

Let $\{a_i\} = \{p_i\} * \{p_i\}$ be the convolution of $\{p_i\}$ with $\{p_i\}$ (so $A(s) = (\phi_X(s))^2$) and note that the expression in parentheses above, is a_{n-1} , so

$$p_n = q a_{n-1}.$$

Multiply this equation by s^n and sum the terms from $n = 2$ to ∞ to see

$$\sum_{n=2}^{\infty} p_n s^n = q s \sum_{n=2}^{\infty} a_{n-1} s^{n-1}.$$

The expression on the left is $\phi_X(s)$, except the first term for $n = 1$ is missing, namely $p_1 s$. Since $p_1 = \mathcal{P}$,

$$\phi_X(s) - \mathcal{P}s = qsA(s) = qs(\phi_X(s))^2,$$

the last equality because $\{a_i\}$ is the convolution of $\{p_i\}$ with itself. This is a quadratic equation (in ϕ_X) with solutions

$$\phi_X(s) = \frac{1 \pm \sqrt{1 - 4\mathcal{P}qs^2}}{2qs}$$

and we can eliminate the “+” solution because it has infinite limit as $s \rightarrow 0$. Evaluate this expression at $s = 1$ and use $q = 1 - \mathcal{P}$, to see that

$$\phi_X(1) = \frac{1 - |1 - 2\mathcal{P}|}{2q}.$$

If $\mathcal{P} < 1/2$, $|1 - 2\mathcal{P}| = (1 - 2\mathcal{P})$ and $\phi_X(1) = \mathcal{P}/q < 1$. If $\mathcal{P} \geq 1/2$, $|1 - 2\mathcal{P}| = 2\mathcal{P} - 1$ and $\phi_X(1) = 1$. Therefore

$$\phi_X(1) = \sum_{n=1}^{\infty} p_n = \begin{cases} \frac{\mathcal{P}}{q} & \text{if } \mathcal{P} < \frac{1}{2} \\ 1 & \text{if } \mathcal{P} \geq \frac{1}{2} \end{cases} \quad (3)$$

$\phi_X(1)$ is the sum of the probabilities that we first lead by 1 after one step, after three steps, after five steps, etc.; i.e., the probability that we at some time get a lead of one over the opponent. According to (3) if the game is biased against us (i.e., if $\mathcal{P} < 1/2$), this probability is $\mathcal{P}/q < 1$ and there is a positive chance $- 1 - \mathcal{P}/q$ - that we never get one unit ahead **SURPRISE 1!** On the other hand we are sure to eventually get a lead of 1 if the coin is fair or biased in our favor.

Now we look at $E(X)$. If $\mathcal{P} < 1/2$ it is infinite by the previous fact - $X = \infty$ with positive probability - so we may assume $\mathcal{P} \geq 1/2$. Differentiating $\phi_X(s) = (1 - \sqrt{1 - 4\mathcal{P}qs^2})/(2qs)$ with respect to s and evaluating the result at $s = 1$ gives

$$E(X) = \phi'_X(s)|_{s=1} = \frac{1}{2\mathcal{P} - 1}.$$

This is infinite for the fair coin with $\mathcal{P} = 1/2$ and gives **SURPRISE 2:** though we are certain to get a lead of 1, we expect to wait infinitely long for it to happen!

Finally, in the case the coin is fair, we note that the previous observations show that the walk is *recurrent*: We are certain to gain 1. After this occurs, we are certain to gain 1 again, etc., so we eventually get to each positive integer. By symmetry we are also certain to get to every negative integer. In fact these remarks really say that if the coin is fair ($\mathcal{P} = 1/2$), **we visit every integer infinitely many times**. On the other hand, and somewhat paradoxically, we expect an infinite wait to arrive at any integer.