Practice Probs on Differentiation and Integration.

1. You want to learn the value of \( w = \log_e 2 \), the natural logarithm of 2. The method is to use the fact that if \( f(t) = 1/t \), the function

\[ I(a) \equiv \int_1^a f(t) dt \]

is \( \log_e a \), so \( w = I(2) \). We will just write \( I \).

(a) Approximate \( I \) using \( \hat{I}_R \), the composite rectangular rule with \( n = 2 \) subdivisions. Be explicit about how you get the numerical value of the approximation. [\( \text{ans: } \hat{I}_R = 5/6 \)]

(b) Write down an expression for the error of the approximation and then use it to get an upper bound for the error. [\( \text{ans: } \) for \( n = 2 \), \( E_R = -1/(4w^2) \), \( w \in [1, 2] \) so \( |E_R| \leq 1/4 \)]

(c) What is the least number of subdivisions needed to guarantee an error less than \( 5 \times 10^{-3} \). [\( \text{ans: } |E_R| \leq 1/(2n) \). Set r.h.s \( < 5 \times 10^{-3} \) to see \( n > 100 \) is enough]

2. Repeat the above using the composite trapezoidal rule. [\( \text{ans: } \hat{I}_T = 17/24 \); for \( n = 2 \), \( E_T = -1/(24w^3) \), \( w \in [1, 2] \) so \( |E_T| \leq 1/24 \); \( |E_T| \leq 1/(6n^2) \)]. Set r.h.s \( < 5 \times 10^{-3} \) to see \( n > 5 \) is enough]

3. Repeat the above using the composite midpoint rule. [\( \text{ans: } \hat{I}_M = 48/70 \); \( E_M = 1/(48w^3) \), \( w \in [1, 2] \) so \( |E_M| \leq 1/48 \); \( |E_M| \leq 1/(12n^2) \)]. Set r.h.s \( < 5 \times 10^{-3} \) to see \( n > 4 \) is enough]

4. Repeat the above using the composite Simpson rule. [\( \text{ans: } \hat{I}_S = 1747/2520 \); for \( n = 2 \), \( E_S = -24/(2880w^5) \), \( w \in [1, 2] \) so \( |E_T| \leq 1/1920 \); \( |E_S| \leq 24/(2880n^4) \)]. Set r.h.s \( < 5 \times 10^{-3} \) to see \( n > 1 \) is enough]

5. We will derive a new rule to approximate the integral of a function \( g(t) \) over the interval \([-1, 1]\). Write the integral as

\[ J = \int_{-1}^1 g(t) dt \]

and let \( J_A \) denote our approximation.

(a) Choosing collocation points \( x_0 = -h \) and \( x_1 = h \), \( h > 0 \) the stepsize, write down the equation of \( I_1(t) \), Newton’s form of the first degree polynomial that interpolates \( g \) at \( x_0, x_1 \).

(b) Now write down the expression you get by integrating \( I_1(t) \) from \(-1\) to \( 1\). This is \( J_A \).

[\( \text{You should get } J_A = g(-h) + g(h) \). When \( h = \sqrt{3}/3 \) this is the simple Gauss2 rule with error proportional to \( g^{(iv)}(t) \) (in other words the error is zero for cubic polynomials)]

6. What is the symmetric difference approximation to the derivative of \( f(t) = t^3 \) at the point \( t = 1 \), given a stepsize \( h \)? Evaluate this expression at \( h = 1/10 \) and \( h = 1/100 \) and compare this to the entries in the table in the handout on differentiation/integration.