Current Midterm, Oct. 28, 2009 - Some Answers

1. Question 1: (5 pts) $(n)_r = (n) \cdot (n-1) \cdots (n-r+1)$ (falling factorial with r terms).

2. Question 2:

   (a) (5 pts) This is an ordered sample of size 6 (with replacement) from a set of size 7. We can write $S = \{(g_1, \ldots, g_6) : g_i$ is the grade she gets in the i^{th} course (CS111 is first; CS344 is last)$\}$; $|S| = 7^6$ by the cartesian product rule.

   (b) (5 pts) Call this event B. $|B| = 5^7$ (so $P(B) = |B|/|S|$). For each course there are 5 grades of C or better.

   (c) (5 pts) Call this event C. $|C| = 7 \cdot (7)^3$: there are 7 choices for the *common* grade for CS111, CS112, CS113, and 7 choices for each of the other three courses.

   (d) (6 pts) Call this event D. $|D| = \binom{6}{2}(6)^4$: there are $\binom{6}{2}$ choices for the two courses that will get B^+ grades and then 6 choices for the (non-B^+) grade for *each* of the other four courses.

   (e) (6 pts) Call this event E. We compute $|E^c|$, the size of the complement, which is the event all grades are the same; $|E^c| = 7$ as there are only 7 grades.

   (f) (6 pts) They are independent if and only if $P_C(D) = P(D)$. To check this we have to compute $P(C \cap D)$. An outcome is in $C \cap D$ if CS111, CS112, and CS113 have the same grades and there are exactly two B^+ grades among the six courses. This means that the common grade of CS111-CS113 *cannot be* B^+. So $|C \cap D| = 6 \cdot 6 \cdot 3$; six choices for the non-B^+ grade of the first three courses, six choices for the non-B^+ grade for the last three courses, and three for which of the last three courses would not get the B^+'s. Finally $P_C(D) = P(D)$ only if

   $$\frac{|C \cap D|}{|C|} = \frac{|D|}{|S|}$$

   and the previous parts show the left side to be smaller than the right

   (g) (6 pts) "neither C nor D" is $C^c \cap D^c = (C \cup D)^c$, so the prob. is $1 - P(C \cup D) = 1 - [P(C) + P(D) - P(C \cap D)]$, all of which have already been computed.

   (h) (7 pts) Call this event H. $|H| = \binom{7}{2} \cdot [\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5}]$: we choose the two grades $x < y$ in one of $\binom{7}{2}$ ways. Then we either get one x and five y, or two x and four y, or three x and three y, or four x and two y, or finally five x and one y. The expression in square brackets is $2^6 - 2$, the number of subsets of a six element set less two (the empty set and the whole set).

   (i) (10 pts) There are $(7)_6$ ways to have all grades different.

   There are $\binom{7}{3}$ ways to have the first three grades increase: every choice of three *distinct* grades can be written uniquely from lowest to highest; on the other hand increasing grades must be distinct, so these two sets are the same. Therefore the probability that the first three grades increase is $\frac{\binom{7}{3}7^3}{7^6}$.

3. Question 4.

   (a) (8 pts) There are (at least) two ways to approach this:

i. Partition experiment: the sample space is the splitting into three groups (ordered), four students (unordered) to each group. $|S| = 12!/(4!)^3$. Let A be the event the cousins are in the same group and B the event they are each in a different group. To count $|A|$, there are three ways to choose *which* group they may all belong to and 9 ways to choose the other student who is also in that group. The remaining 8 students are distributed to two groups in $\binom{8}{4}$ ways, so $|A| = 27 \cdot \binom{8}{4}$. There are 3! ways to place the cousins, one to a group and for each, $\binom{9}{3}$ ways to complete the first group and $\binom{6}{3}$ ways to complete the second, so $|B| = 6\binom{9}{3}\binom{6}{3}$.

ii. The second approach is to regard the experiment as an ordered sample of the 12 students, without replacement: the first four is group one, the second four is group two, etc. $|S| = 12!$. Under this approach the youngest cousin has 12 possible placements. The others may be placed in his group of four in $3 \cdot 2$ ways and the last spot in that group of four may get any of the 9 non-cousins, so $|A| = 12 \cdot 6 \cdot 9 \cdot 8!$; $P(A)$ is the same in both methods.

(b) (7 pts) The experiment is an unordered sample of size 14 from a set of size 27, so $|S| = \binom{27}{14}$. For the event of interest, exactly four women may be chosen in $\binom{11}{4} \cdot \binom{16}{10}$ different ways.

As above, note that for $j = 0, \ldots, 11$, we can have j women among the 14 chosen students in exactly $\binom{11}{j} \cdot \binom{16}{14-j}$ different ways. Therefore the event "at least 4 chosen" has size

$$\sum_{j=4}^{11} \binom{11}{j} \cdot \binom{16}{14-j}$$

but it may be easier to write down the size of the complement (sum from zero to three).