Old Midterm, Oct. 21, 2000 - Some Answers

1. Question 1:

   (a) This is an ordered sample of size 5 (with replacement) from a set of size 52. We can write $S = \{(c_1, \ldots, c_5) : c_i \text{ is the card you get on the } i^{th} \text{ pick}\}$; $|S| = (52)^5$ - by the cartesian product rule.

   (b) $|A| = 4^5$ so $P(A) = |A|/|S|$. (On each pick, there are 4 ways to get a King)

   (c) $|B| = 13|A|$. King is one of the 13 possible values. Each possibility has the same number of outcomes as A.

   (d) By the conditional probability formula $P_B(A) = (|A \cap B|/|B| = 1/13$. They are NOT independent because $P(A) = (1/13)^5$.

   (e) $|D| = \binom{13}{2} \cdot 4^2 \cdot 52^3$. This is tricky. There are $\binom{13}{2}$ distinct pairs of values $x < y$ for the first two cards. This covers all the ways in which the first cards value (which we take as x), is smaller than the second cards value, y; then there are 16 choices for the suits of x and y, and finally $(52)^3$ possibilities for the last 3 cards.

   (f) If E is "full house", $|E| = 13 \cdot \binom{5}{3} \cdot 12 \cdot 4^5$. There are 13 choices for the value of x, $\binom{5}{3}$ choices for which $3c_i$'s are x, 12 choices for y and 4^5 choices of suits.

   (g) If F is "one pair", $|F| = 13 \cdot \binom{5}{2} \cdot \binom{12}{3} \cdot 3! \cdot 4^5$.

2. Question 2.

   (a) This is the partition experiment: the 12 people are partitioned into three sets, each of size four (one for each trip). Therefore $|S| = \binom{12}{4}\binom{8}{4}$. The event $A = \{\text{the thin ones go in the first trip}\}$ has size $\binom{8}{4}$ (the number of different ways to do the remaining two trips); similarly the number of outcomes where the thin ones go in the second trip is $\binom{8}{4}$, as is the number of outcomes where they go in the third. So $3/\binom{12}{4}$ is the probability. [Alternatively we can look at this as a permutation of the 12 people of three types, thin, fat, just right. We can distinguish between the types, but not the people of the *same* type. There are $\binom{12}{4}\binom{8}{4}$ distinguishable permutations. For each permutation the first four people go in trip one, the next four in trip two, the last in trip three. The thin ones travel together in $3\binom{8}{4}$ of them, as before]

   (b) 6/14 - draw them.

   (c) This is the partition experiment: bridge (model 2), so $|S| = 52!/(13!)^4$. The number of outcomes where eacch hand has an Ace and a King is $(4!)^2\binom{44}{11}\binom{33}{11}\binom{22}{11}$ (there are 4! ways to give one Ace to each hand; same for one King to each; the last expression distributes the 44 non-Ace, non-Kings, 11 to each hand).

   (d) 71/120 using inclusion/exclusion. Let A_i be the event that person i gets their own hat and $A = \bigcup_{i=1}^{3} A_i$. We want the probability of A^c which is $1 - P(A) = 1 - [S_1 - S_2 + S_3]$. Since $P(A_i) = 5!/6!$, $S_1 = 3(1/6)$; since $P(A_i \cap A_j) = 4!/6!$, $S_2 = 3 \cdot (1/30)$, and since $P(A_i \cap A_j \cap A_k) = 3!/6!$, $S_3 = 1/(6)_3$, and $P(A) = 1 - [3/6 - 3/(6 \cdot 5) + 1/(6 \cdot 5 \cdot 4)]$.