CS442 **March 27 2008**

TEST 1

Instructions: Do all your work in the blue exam books. Please write your answers IN THE GIVEN ORDER, though you may solve problems in any order. There is no need to reduce answers to simplest terms. You may use ONE PAGE OF PREPARED NOTES, but all work must be your own. Show *ALL* your work. You will get *little* or *no* credit for an unexplained answer. The value of each question appears in parentheses. Use this as a guide in allocating your time. There are 80 points, and you have 80 minutes.

1. In this question we have a coin with $\mathcal{P}$ as the probability of HEAD (not necessarily 1/2). Ultimately we want to use it to simulate the toss of a fair die. To start though, we want to use the coin to generate the values $1, 2, 3$, with equal probability. Three methods are proposed:

   - Method A tosses the coin twice and outputs 1 plus the number of heads;
   - Method B tosses the coin three times. If there is *only ONE* HEAD produced, and if it appeared on the i^{th} toss, the value i is output, otherwise, no output.
   - Method C tosses the coin twice. If HH it outputs 1, if HT it outputs 2, if TH it outputs 3, and if TT it gives no output.

   (a) (8 points) For which values of $\mathcal{P}$, if any, does each method produce the numbers $1, 2,$ and 3 with equal probabilities?

   P(A outputs 1)= P(TT) $= (1 - \mathcal{P})^2$
   P(A outputs 2)= P(TH or HT) $= 2\mathcal{P}(1 - \mathcal{P})$
   P(A outputs 3)= P(HH) $= \mathcal{P}^2$
   For no value of $\mathcal{P}$ these quantities are equal.

   P(B outputs 1)= P(B outputs 2)= P(B outputs 3)= $\mathcal{P}(1 - \mathcal{P})^2$
   So for any value of $\mathcal{P} \in [0, 1]$ this method works.

   P(C outputs 1)= P(HH) $= \mathcal{P}^2$
   P(C outputs 2)= P(HT) $= \mathcal{P}(1 - \mathcal{P})$
   P(C outputs 3)= P(TH) $= (1 - \mathcal{P})\mathcal{P}^2$
   These probabilities are equal when $\mathcal{P} = 1/2$.

   (b) (7 points) For methods B and C, compute the expected number of tosses needed to produce an output, as a function of $\mathcal{P}$.

   For Method B, the number of trials is a geometric random variable with success probability $3\mathcal{P}(1 - \mathcal{P})^2$. So the expected number of trials is given by $1/(3\mathcal{P}(1 - \mathcal{P})^2)$. Since we do three coin tosses per trial, the expected number of coin tosses is $1/(\mathcal{P}(1 - \mathcal{P})^2)$.

   For Method C, the expected number of trials, by the same analysis as above is 2, and the expected number of tosses is 4.

(c) (5 points) Compute the variance of the number of tosses method B needs to produce an output, as a function of $\mathcal{P}$. Whats the most favorable value for $\mathcal{P}$?

The variance of a geometric random variable is given by: $\frac{(1-p)}{p^2}$, when p is the probability of success.
So in our case the variance for the number of trials, is $\frac{1-3(\mathcal{P}(1-\mathcal{P})^2)}{(3\mathcal{P}(1-\mathcal{P})^2)^2}$.
The variance of number of tosses is 9 times the variance of number of trials.
The mean $1/(\mathcal{P}(1-\mathcal{P})^2)$, is minimized at $\mathcal{P} = 1/3$. Note that this is also the minima for the variance. So the most favorable value of $\mathcal{P}$ is 1/3.

(d) (5 points) Explain how you could simulate a fair dice with a biased coin.

Like method B, we toss the coin 6 times. If there is *only ONE* HEAD produced, and if it appeared on the i^{th} toss, the value i is output, otherwise, no output.

(e) (5 points) Now suppose you have access to a (large, say $1,000,000$) set of biased coins. Each has its own distinct value for $\mathcal{P} \neq 1/2$. You are only allowed to toss any coin at most two times. Can you generate the numbers $1, 2$, and 3 with equal probabilities? Show how to do it or explain why its not possible.

Here is one possible approach: Toss a coin twice. If it comes up HH or TT, no output; otherwise this coin gives a (fair) HEAD if the tosses were HT, and a (fair) TAIL if the tosses were TH. In all cases the coin is discarded after its two tosses.

When three coins have given outputs, if there was exactly one (fair) HEAD and it was from the $i-th$ (fair) coin, output i; else disregard these three coins and repeat. Thus you might be able to generate the random digits with your set of coins. However there is a positive probability that you exhaust the finite set of coins before this happens so you must conclude that it is impossible to do, with certainty.

This would be the case for ANY method you used with this set of coins and the given rules, because you would have to wait for a coin to be useful in producing an equally likely output, and this random variable has an infinite range.

2. (5 pts) Give some constructive criticism of the course: (i) what is bad and should be improved? (ii) what is good and should be continued? (iii) what is missing and should be added?

3. (20 pts) This question deals with random permutations. The probability space is $S = \{\underline{\pi} = (\pi_1, \ldots, \pi_n)\}$ of permutations of $1, \ldots, n$ under equally likely probability. Here $n = 2k$ is even.

(a) Let N be the random variable that counts $|\{j : \pi_j = j\}|$. Compute the probability that $N + 3 = n$ and explain how you did it.

We have to compute the probability of the event that all but 3 elements go to their identity positions.
For fixed 3 elements, say $\{1, 2, 3\}$ there are two possibilities that this event occur. Namely, $\pi_1 = 2, \pi_2 = 3, \pi_3 = 1$ and $\pi_1 = 3, \pi_2 = 1, \pi_3 = 2$. So the probability that we get $\pi_j = j$ for each $j = 1 \ldots n$ and $j \notin \{1, 2, 3\}$ is $2/n!$.

Since we have $\binom{n}{3}$ such mutually exclusive events. The probability that any of them occurs is $\binom{n}{3}\frac{2}{n!}$

(b) Let B be the event that that $\pi_j \leq k$, $j = 1, \ldots, k$. Find $P(B)$ and explain your answer.

We have to find the probability that the smaller k numbers go to lower k position and larger k goes to higher k position. Since there are $k!$ ways to permute the smaller k numbers within the lower half and the same is true for larger k numbers. So $P(B) = \frac{k!k!}{n!}$.

(c) Let C be the event that for each $j = 1, \ldots, k$, $\pi_{2j} = 2j$ and $\pi_{2j-1} \neq 2j - 1$. Are B and C independent?

We will show that $P(B) \neq P(B|C)$.

For a simple example let $k = 2$.
The event $B = \{1234, 1243, 2134, 2143\}$
and the event $C = \{3214\}$.
Obviously in this case $P(B) = \frac{1}{6} \neq P(B|C) = 0$

In general, the event $B \cap C$ has permutations with following properties. All even numbers are fixed points, smaller $k/2$ odd numbers are in the lower half and larger $k/2$ odd numbers are in the upper half. Furthermore, smaller $k/2$ odd numbers make a derangement within the lower half and larger odd numbers make derangement in upper half.

Recall that a derangement is a permutation π, such that there is no fixed point, i.e $\pi_i \neq i$ for all i. Let d_i be the number of possible derangements in permutations of i numbers. Clearly by above definitions,

$P(B \cap C) = \frac{d_{k/2} d_{k/2}}{n!}$ and $P(C) = \frac{d_k}{n!}$.

So $P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{d_{k/2} d_{k/2}}{d_k}$

In order to show that $P(B) \neq P(B|C)$, we show that $\frac{P(B|C)}{P(B)} \neq 1$. This can be proven by simple calculation and using the fact that for large n, $\frac{d_n}{n!}$ approaches $\frac{1}{e}$.

(d) **(cycles of length 2, etc.)** If $\pi_i = j \neq i$ and $\pi_j = i$, we say that i and j form a cycle of length 2 in $\underline{\pi}$. Compute $E(N)$, the expected number of cycles of length 2 in $\underline{\pi}$. What about cycles of length 3? What about the total number of cycles in $\underline{\pi}$?

Lets define $\binom{n}{2}$ random variables X_{ij} for $i < j$, which takes value 1 if i and j make a cycle of length 2, i.e. $\pi_i = j$ and $\pi_j = i$ and is 0 otherwise. $P(X_{ij} = 1) = \frac{1}{n(n-1)}$. Hence the $E(X_{ij}) = \frac{1}{n(n-1)}$.
Let N_2 be the number of cycles of length 2 in a permutation. Clearly $N_2 = \sum_{i<j} X_{ij}$, So $E(N_2) = E(\sum_{i<j} X_{ij}) = \sum_{i<j} E(X_{ij}) = \sum_{i<j} \frac{1}{n(n-1)} = \binom{n}{2}\frac{1}{n(n-1)} = \frac{1}{2}$.

The expectation of the number of k cycles, N_k can be found similarly, by having an indicator random variable $X_{i_1,i_2,\ldots,i_k} = 1$ if $i_1, i_2, \ldots, i_k$ make a k cycle, for example $\pi_{i_1} = i_2, \pi_{i_2} = i_3, \ldots, \pi_{i_k} = i_1$ and 0 otherwise. Note that there are $(k-1)!$ different cycles that can be made from the same k numbers. Clearly $E(X_{i_1,i_2,\ldots,i_k}) = \frac{(k-1)!}{n(n-1)\ldots(n-k+1)}$. Similar to the above calculation, by linearity of expectation, $E(N_k) = \binom{n}{k}\frac{(k-1)!}{n(n-1)\ldots(n-k+1)} = \frac{1}{k}$.

The total number of cycles N is just the sum of $N_1, N_2, \ldots, N_n$ and expectation is found by linearity of expectation, which is H_n, the nth harmonic number.

4. (25 pts) X and Y are *independent* random variables on the same probability space. The means satisfy $E(X) = E(Y) = 1$ and the variances satisfy $V(X) = V(Y) = 2$. For each of the following statements, decide whether it is TRUE or FALSE ("TRUE" means that the statement must always be true for random variables satisfying the given conditions). If you say TRUE, give a convincing reason. If you say FALSE, give a counter-example.

   (a) $P(X \geq 2) \leq 1/2$
   FALSE: Consider random variable X such that $P(X = 2) = 2/3$ and $P(X = -1) = 1/3$. X satisfies all conditions.

   (b) $P(X = 1) < 1$
   TRUE: Because otherwise the variance of X would be 0.

   (c) $V(X + 2Y + 5) = 10$
   TRUE: By Independence $V(X + 2Y + 5) = V(X) + 4V(Y) = 2 + 4(2) = 10$.

   (d) $E(1/X) = 1$
   FALSE: By definition $E(1/X) = \sum_{x \in domain(X)}(1/xP(X = x))$. Consider $P(X = 1 + \sqrt{2}) = 1/2$ and $P(X = 1 - \sqrt{2}) = 1/2$. This X satisfies the mean and variance property. But in this case $E(1/X)$ is -1.

   (e) $E(XY) = 1$
   TRUE: By independence $E(XY) = E(X)E(Y)$, Since $E(X) = E(Y) = 1$ we have $E(XY) = 1$.