CS442 **March 1, 2005**

TEST 1

Instructions: Do all your work in the blue exam books. Please write your answers IN THE GIVEN ORDER, though you may solve problems in any order. There is no need to reduce answers to simplest terms. You may use one page of prepared notes, but all work must be your own. Show *ALL* your work. You will get *little* or *no* credit for an unexplained answer. The value of each question appears in parentheses. Use this as a guide in allocating your time. There are 80 points, and you have 80 minutes.

1. (30 pts) X and Y are random variables on the same probability space. X has mean $E(X) = 2$, variance $V(X) = 9$, and $P(X > 10) = 0$. Y has mean $E(Y) = 3$, and $E(XY) = 6$. For each of the following statements, decide whether it is TRUE or FALSE ("TRUE" means that the statement must always be true for random variables satisfying the given conditions). If you say TRUE, give a convincing reason. If you say FALSE, give a counter-example.
   (a) $P(Y = 3) < 1$
   (b) $P(X \geq 1) \geq \frac{1}{10}$
   (c) $V(X + Y) = 9 + V(Y)$
   (d) $P(X \geq 8) \leq 1/4$
   (e) $P(X \geq 5) \leq 2/5$
   (f) X and Y are independent.
2. (20 pts) This question deals with random permutations. The probability space is $S = \{\underline{\pi} = (\pi_1, \ldots, \pi_n)\}$ of permutations of $1, \ldots, n$ under equally likely probability. Here $n = 2k$ is even.
   (a) Let A be the event that the even and odd values alternate (if π_i is odd, then π_{i+1} is even; if π_i is even, then π_{i+1} is odd, $i < n$). Find $P(A)$ and explain how you did it.
   (b) Let B be the event that the odd π_i appear in increasing order ($\pi_i < \pi_j$ if both are odd and $i < j$). Find $P(B)$ and explain how you did it. Explain *in English* how you would generate a permutation in B, each one equally likely (extra credit if you do it in linear time).
   (c) Let C be the event that $\underline{\pi}$ has two cycles of length $n/2$, the odd numbers in one cycle and the even numbers in the other. Find $P(C)$ and explain how you did it.
   (d) Let D be the event that there are exactly $n/2$ cycles of length one, and one cycle of length $n/2$. Find $P(D)$ and explain how you did it. Are C and D independent? Explain.
3. (5 pts) Give some constructive criticism of the course: (i) what is bad and should be improved? (ii) what is good and should be continued? (iii) what is missing and should be added?

4. (15 pts) Given a set S of $n = 2k+1$ real inputs $a_i, i = 1\dots, n$, all distinct, the task is to return an element $x \in S$ that is smaller than the median $\mu \in S$; i.e., the $k+1^{st}$ smallest element of S. An obvious algorithm is to take ANY $k+2$ elements of S and return their minimum. The cost is $k+1$ comparisons. Carefully describe a probabilistic algorithm for this task which has running time $o(n)$ as $n \to \infty$ and which returns a correct answer with probability at least $1-\varepsilon$, where $\varepsilon > 0$ a given constant (the faster the algorithm, the better). Make it clear what the running time of your algorithm is (measured in terms of (i) the number of comparisons and (ii) the number of calls to UNIF). [(*) If you wish, and if you have extra time, comment on the possibility of a better deterministic algorithm - no penalty if you don't]

5. (10 pts) This question concerns Karger's basic min-cut algorithm (ALG 0): We start with a connected graph $G = (V, E)$ having n vertices and $m \geq n-1$ undirected edges (each of weight one. While our (multi)graph has more than 2 vertices we contract a randomly chosen edge. At the end we output the remaining edges that join the final two vertices; they are a cut in the original G. In this problem G is a cycle of length $n-1$ plus a single edge attached to it (so G has n vertices and edges which we take as $\overline{v_1v_2}, \overline{v_2v_3}, \dots, \overline{v_{n-2}v_{n-1}}, \overline{v_{n-1}v_1}$ and $\overline{v_1v_n}$.

   (a) Draw G. What is a min-cut for G?

   (b) What is the probability that ALG 0 will find a min-cut? Explain your answer (you should describe how G contracts to the final min-cut during a successful contraction sequence, showing the last two contraction steps).