CS442 **March 27 2008**

TEST 1

<u>**Instructions:**</u> Do all your work in the blue exam books. Please write your answers IN THE GIVEN ORDER, though you may solve problems in any order. There is no need to reduce answers to simplest terms. You may use ONE PAGE OF PREPARED NOTES, but all work must be your own. Show *ALL* your work. You will get *little* or *no* credit for an unexplained answer. The value of each question appears in parentheses. Use this as a guide in allocating your time. There are 80 points, and you have 80 minutes.

1. In this question we have a coin with $\mathcal{P}$ as the probability of HEAD (not necessarily 1/2). Ultimately we want to use it to simulate the toss of a fair die. To start though, we want to use the coin to generate the values $1, 2, 3$, with equal probability. Three methods are proposed:

   - Method A tosses the coin twice and outputs 1 plus the number of heads;
   - Method B tosses the coin three times. If there is *only ONE* HEAD produced, and if it appeared on the i^{th} toss, the value i is output, otherwise, no output.
   - Method C tosses the coin twice. If HH it outputs 1, if HT it outputs 2, if TH it outputs 3, and if TT it gives no output.

   (a) (8 points) For which values of $\mathcal{P}$, if any, does each method produce the numbers $1, 2$, and 3 with equal probabilities?

   (b) (7 points) For methods B and C, compute the expected number of tosses needed to produce an output, as a function of $\mathcal{P}$.

   (c) (5 points) Compute the variance of the number of tosses method B needs to produce an output, as a function of $\mathcal{P}$. Whats the most favorable value for $\mathcal{P}$?

   (d) (5 points) Explain how you could simulate a fair dice with a biased coin.

   (e) (5 points) Now suppose you have access to a (large, say $1,000,000$) set of biased coins. Each has its own distinct value for $\mathcal{P} \neq 1/2$. You are only allowed to toss any coin at most two times. Can you generate the numbers $1, 2$, and 3 with equal probabilities? Show how to do it or explain why its not possible.

2. (5 pts) Give some constructive criticism of the course: (i) what is bad and should be improved? (ii) what is good and should be continued? (iii) what is missing and should be added?

3. (20 pts) This question deals with random permutations. The probability space is $S = \{\underline{\pi} = (\pi_1, \ldots, \pi_n)\}$ of permutations of $1, \ldots, n$ under equally likely probability. Here $n = 2k$ is even.

   (a) Let N be the random variable that counts $|\{j : \pi_j = j\}|$. Compute the probability that $N + 3 = n$ and explain how you did it.

   (b) Let B be the event that that $\pi_j \leq k$, $j = 1, \ldots, k$. Find $P(B)$ and explain your answer.

(c) Let C be the event that for each $j = 1, \ldots, k$, $\pi_{2j} = 2j$ and $\pi_{2j-1} \neq 2j - 1$. Are B and C independent?

(d) **(cycles of length 2, etc.)** If $\pi_i = j \neq i$ and $\pi_j = i$, we say that i and j form a cycle of length 2 in $\underline{\pi}$. Compute $E(N)$, the expected number of cycles of length 2 in $\underline{\pi}$. What about cycles of length 3? What about the total number of cycles in $\underline{\pi}$?

4. (25 pts) X and Y are *independent* random variables on the same probability space. The means satisfy $E(X) = E(Y) = 1$ and the variances satisfy $V(X) = V(Y) = 2$. For each of the following statements, decide whether it is TRUE or FALSE ("TRUE" means that the statement must always be true for random variables satisfying the given conditions). If you say TRUE, give a convincing reason. If you say FALSE, give a counter-example.

   (a) $P(X \geq 2) \leq 1/2$

   (b) $P(X = 1) < 1$

   (c) $V(X + 2Y + 5) = 10$

   (d) $E(1/X) = 1$

   (e) $E(XY) = 1$