CS206 **HW2** **Feb. 5, 2012**

Write up convincing answers to the questions marked by an asterisk (*). Some of the unstarred questions will be worked in the recitations. (**) indicates a more challenging problem. It does *NOT* have to be handed in, but if you do, we will check your work and give feedback. Your solutions are due in class on Feb. 20, 2012.

1. Let A,B,C, and D be events in a sample space S. Express the following events - described in English - using the operators $\cup$, $\cap$, $\setminus$, and complement.
   (a) Exactly one of A, B, or C occur.
   (b) (*) A occurs but neither B nor C do.
   (c) Three or more of the events occur.
   (d) (*) Exactly two of the events occur.
   (e) (*) Exactly one of A, B, C occur, as well as D. Repeat, now excluding D but requiring exactly one of the other three.

2. Do the following:
   (a) If $P(A) = 1/3$, $P(B) = 1/2$, and $P(A \cup B) = 3/4$ compute $P(A \cap B)$, $P(A^c \cup B^c)$, and $P(A^c \cap B)$.
   (b) (*) An experiment has two outcomes, one with probability $1 - p$, the other with probability $2p^2$. What is p? Explain your answer.
   (c) (*) Prove $P[(A \cap B^c) \cup (A^c \cap B)] = P(A) + P(B) - 2P(A \cap B)$. Describe this event in English.
   (d) Use equally likely probability measure and compute the probability of the events in questions 4a and 4b, Homework 1. The TA will do the other parts of question 4, HW1.
   (e) (*) As above, find $P_A(B)$, the conditional probability of B, given A. The TA will do the unstarred parts.

3. (S, P) is the probability space of tossing two dice (one red and one blue) under equally likely probability.
   (a) (*) Let $A = \{\text{red die is } 3, 4, \text{ or } 5\}$, $B = \{\text{blue die is 1 or 2}\}$, and $C = \{\text{sum is 7}\}$. Are these events pairwise independent?, Are they mutually independent?
   (b) Let $A = \{\text{red die is odd}\}$, $B = \{\text{blue die is odd}\}$, and $C = \{\text{the sum is odd}\}$. Compute $P_A(B)$, $P_A(C)$, $P_B(C)$. Are they pairwise independent? Explain. Are they mutually independent? Explain.
   (c) (**) Let $A = \{\text{sum is 7}\}$ and $B = \{\text{there is at least one 6}\}$. Compute $P(A)$ and $P_B(A)$. Here is an apparent paradox: compute $P_{C_x}(A)$ where, for $x = 1, 2, 3, 4, 5$, or 6, $C_x = \{\text{there is at least one } x\}$. Since you know that some C_x always occcurs, how can $P(A) \neq P_{C_x}(A)$? Discuss whether this makes sense or not. How do you reconcile it?

4. A computer has printer (P), disk (D) and terminal (T) outputs. Sixty percent of all output characters are on D, thirty percent on P, and the rest on T. The error rate for D is 1/2000, for P it is 2/1000, and for T it is 1/1000. The experiment $\mathcal{E}$ is that a character is output and we observe (i) which type of device made that output and (ii) whether the character was correct. Write down the sample space. What is the probability the character was written on the disk, given $A = \{\text{it was incorrect}\}$? [Bayes rule].

5. (*) A box contains 100 balls. 20 are red, 30 are green, and the rest are yellow. 3/4 of the red balls are small (the rest are big), 2/3 of the green balls are small, and 1/2 of the yellow balls are small. The experiment is to choose a ball at random and to observe its color and its size.

   (a) Describe the probability space for this experiment.

   (b) What is the probability of the event $A = \{\text{a small ball is chosen}\}$?

   (c) You are told that A occurs. What is $P_A(\text{red})$?

6. (**) There are 4 envelopes, one of which contains 100 dollars, the other 3 being empty. You take one of the envelopes at random.

   (a) What is the probability that when you open it, it will contain 100 dollars?

   (b) Now, before you open your envelope, somebody opens one of the other 3 and shows you that it is empty. You are now offered the choice to (i) keep your original envelope or (ii) change to one of the remaining 2. What is the probability you win 100 dollars if you do (i)? What is the probability if you do (ii)? Explain in detail.

7. (On independence and nuances).

   (a) (*) Decide if A and B can be independent if they are mutually exclusive, and explain your answer.

   (b) Decide if A and B can be independent if $A \subseteq B$ and explain your answer. You may assume both events have probability larger than zero abd smaller than one.

   (c) Assume that A, B, C are pairwise independent events. Does it necessarily follow that $A \cup B$ and C are independent? If YES, try to show why. If NO, give an example where it fails.

   (d) (*) Repeat the above question, now assuming that A, B, C are mutually independent.

   (e) (**) When are the events $A \cap B$ and C independent? (a) Always; (b) never? (c) sometimes. If you say (c) try to characterize when it occurs.

   (f) Describe a probability space (S, P) and events in it (the simpler, the better), where the following hold:

      i. (*) The events A_1, A_2, A_3 are not pairwise independent although A_1 and A_2 are independent and A_1 and A_3 are independent.

      ii. In the family of events $A_1, \ldots, A_4$, A_1, A_2, A_3 are pairwise independent and A_1, A_2, A_4 are pairwise independent, but A_3 and A_4 are not independent.

      iii. (**) Events $A_1, \ldots, A_4$ are not 3-wise independent although A_1, A_2, A_3, A_1, A_2, A_4, and A_1, A_3, A_4 are each 3-wise independent.

NOW, SOME COUNTING PROBLEMS:

8. (*) A man is buying a present. He will choose a tie or a shirt. There are 3 ties and 2 shirts to choose from. How many choices has he if he buys one item? How many if he buys one tie and one shirt? What if he buys any two items?

9. How many integers are there larger than one million, smaller than ten million, and having no consecutive digits the same? Explain your counting.

10. (*) A town has 17,910 people. Show that at least two have the same set of 3 initials.

11. (*) A domino is made by gluing two square pieces together; each square piece has a number from 1 to 9. Order is not important - the domino [2 3] may be used interchangeably with [3 2] in the playing of the game. How many different dominos are there?

12. How many ways can 4 men and 4 women form (heterosexual) couples? How many ways can they stand in a row, alternating sexes? How many ways can they stand in a circle, alternating sexes?

13. (*) Sample twice from $T = \{1, \ldots, 9\}$ without replacement. Find the probability that an odd digit will be selected (a) on the first choice, (b) on the second choice, and (c) on both choices. Explain your answer.

14. How many of the permutations of the first nine positive integers have all the even numbers before any of the odd ones?

15. (*) Use Stirlings's approximation to estimate 30!. The truth is $265,252,859,812,268,935,315,188,480,000,000$.

16. How many straight lines can be drawn through six points A, B, C, D, E, F, no three of which are collinear?

17. (*) Ten basketball players meet for a game. In how many ways can they be divided into two teams of five each? Explain your answer.

18. Use Stirlings's approximation to show that $\binom{2n}{n} \sim 4^n/\sqrt{\pi n}$; i.e., the ratio of the two expressions converges to one.

19. If n balls are randomly placed into n boxes, find the probability that exactly one box is empty. Explain your reasoning.

20. (**) If $n + 1$ balls are randomly placed into n boxes, find the probability that exactly one box is empty. Explain your reasoning.