HW3 - Some Solutions

- **Question 1b**: \(|S| = (\binom{13}{3})\) (the number of different bridge hands a player could see). \(|A| = (\binom{10}{2})\) (there are 39 non-hearts) so \(P(A) = |A|/|S| = .01279\ldots\). \(|B| = (\binom{8}{13})\) and \(P(B) = .30338\ldots\). Finally \(P_A(B) = |A \cap B|/|A| = (\binom{3}{13})/(\binom{3}{13})\) (36 non-hearts that are also non-kings) (prob = 0.28495\ldots).

- **Question 1c**: Let \(C = \{\text{no spades}\}\) and \(D = \{\text{no card} > 9\}\). \(P(C) = |C|/|S| = (\binom{39}{13})/(\binom{52}{13})\) and \(P(D) = (\binom{32}{13})/(\binom{52}{13})\) (there are 32 cards with values 2-9). “Neither” is \(C^c \cap D^c = (C \cup D)^c\). Therefore

\[
P((C \cup D)^c) = 1 - [P(C) + P(D) - P(C \cap D)]
\]

and we finish using \(|C \cap D| = (\binom{24}{13})\) (there are 24 non-spades from two to nine, inclusive).

- **Question 1e**: Model 2 probabilities agree with those from Model 1. I illustrate this by counting the size of the event \(B = \{\text{West gets NO aces}\}\). West could be dealt any of the \(\binom{48}{13}\) possible hands that have NO aces. Let's reserve the 13 cards in one of these hands for later distribution to West. Now North can be given any of the \(\binom{39}{13}\) possible hands excluding the cards reserved for West. Then East may be given one of the \(\binom{26}{13}\) hands excluding West's reserved cards and the 13 given to North. Finally we give West his 13 reserved cards and South the remaining 13 cards. There are \(\binom{48}{13}(\binom{39}{13})(\binom{26}{13})\) different outcomes to this process, by the cartesian product principle, and there are no other possible hands in \(B\). This method of counting shows that an event model one for bridge has the same probability for any of the four players under model two.

- **3b**: First, \(E\) is ”pick a key”. \(S\) is the key you get and it is THE good key with prob = \(1/|S| = 1/7\).

Now, the experiment is to pick a key; try it, and then repeat. \(S = \{(k_1, k_2) : k_i\text{ is the key you got on the }i^{th}\text{ pick}\}\), so \(|S| = 7^2\). \(B\) is the event the first pick was bad and the second was good, so \(|B| = 6 \cdot 1\) and \(P(B) = 6/49\). Continuing this one step further, you first get in on the third try with probability \(6^2/7^3\). For 3**, he gets in first on try 5 with probability \(6^4/7^5\) and in general on try \(j\) with probability \((6/7)^j(1/7)\).

- **4c**: The random drawing must get THREE of YOUR five pool 1 numbers and TWO of your non-pool-1 numbers. In addition its pool 2 number must agree with YOURS. The probability is

\[
\frac{\binom{3}{5}(\binom{70}{2})}{\binom{5}{2} \cdot 15} = .0000032825519,
\]

quite close to what they listed.

- **Question 5b,c**: Clearly \(P(A) = P(B) = P(C) = 5!/7! = 1/42\). “None” is the event \(D = A^c \cap B^c \cap C^c = (A \cup B \cup C)^c\), so by Inclusion/Exclusion,

\[
P(D) = 1 - [P(A \cup B \cup C)] = 1 - [S_1 - S_2 + S_3],
\]

where \(S_1 = P(A) + P(B) + P(C) = 3(5!/7!)\), \(S_2 = P(A \cap B) + P(A \cap C) + P(B \cap C) = 3(3!/7!)\), and \(S_3 = P(A \cap B \cap C) = 1/7!\). ANS: 671/720.

“ALL” is the outcome \((h_1, \ldots, h_7) = (2, 1, 5, 6, 3, 4, 7)\), the hat person \(i\) gets, so it has probability \(1/7!\).
• 7: The sample space for choosing five shoes from a set of twelve pairs of shoes is all subsets of five shoes, and its size is $|S| = \binom{24}{5}$. The event $A$ of NO pseudo-pair has all five choices being LEFT shoes or all five being RIGHT shoes, so $P(A) = 2\binom{12}{5}/\binom{24}{5} = 0.0307267$. The event $B$ of no PAIR chooses 5 of the pairs and from each, one of the two shoes, so $|B| = \binom{12}{5} \times 2^5$ and $P(B) = 0.59627$. 

• 9c,d: $S = \{ (A_1, \ldots, A_{13}) \},$ the $A_i$ a partition of the 52 cards into 13 (ordered) subsets (each subsets elements are unordered) of four cards each}. $|S| = \frac{52!}{(4!)^{13}}$. 

$|A| = (13!)^4$ - there are 13! different ways each player gets a spade, 13! ways each gets a heart, etc.

$|B| = 13!$ - the permutations of the 13 values.

$P_A(B) = \frac{|A \cap B|}{|A|} = \frac{|B|}{|A|},$ as $B \subseteq A$. For the same reason, $P_B(A) = 1$. 

2