## HW5 - Some Solutions

- **Question 3 (ii):** I will use the fact (Rev. Sheet 2) that in the $n$ hat experiment, the probability that $j$ people get their own hats is $q_{n-j}/j!$, where $q_k = 1 - 1/1! + 1/2! + \cdots + (-1)^k/k!$. Therefore for the 4 hat case when $X$ is the number getting their own hats, $P(X = 0) = q_4 = 9/24$, $P(X = 1) = q_3 = 1/3$, $P(X = 2) = q_2/2! = 1/4$, $P(X = 3) = 0$, and $P(X = 4) = 1/24$. This gives $E(X) = 0 \cdot 9/24 + 1 \cdot 1/3 + 2 \cdot 1/4 + 4cdot1/24 = 1$

- **Question 5:** Let $X_P$ be the indicator of the event that the printer is demanded by at least one of the seven jobs. Then

$$E(X_P) = 0 \cdot \mathrm{Prob(X_P = 0)} + 1 \cdot \mathrm{Prob(X_P = 1)} = 1 - \mathrm{Prob(X_P = 0)} = 1 - (\frac{11}{12})^7.$$

Similarly for the tape we have $E(X_t) = 1 - (5/6)^7$, for the disk we have $E(X_D) = 1 - (7/12)^7$ and for the terminal, $E(X_\tau) = 1 - (2/3)^7$. Since $N$ is the sum of the four indicators,

$$E(N) = 1 - (11/12)^7 + 1 - (5/6)^7 + 1 - (7/12)^7 + 1 - (2/3)^7,$$

and this is $4 - (11/12)^7 - (5/6)^7 - (7/12)^7 - (2/3)^7 = 3.095555937...$

- **Question 6:** Because we are sampling *with* replacement, the experiment is a Bernoulli trial with success probability 1/7 and $W$ is the number of trials needed to get one success, a geometric random variable with $\mathcal{P} = 1/7$. Therefore $E(W) = 7$.

- **Question 7:** We are sampling without replacement from the seven keys, and using product probability. $P(X = 1) = 1/7$ as you must choose the good key right away. If $X = 2$ we must choose a bad key on the first pick (prob = 6/7) and then the good one from the remaining six (prob = 1/6 because the bad key from pick one was not replaced) so the product probability that $X = 2$ is $(6/7)(1/6) = 1/7$. In the same way we show that for each $j \in \{1, \ldots, 7\}$, $\mathrm{Prob}(X = j) = 1/7$. Without replacement is better because (i), $E(X) = 4$ and (ii), you are sure to get in after seven tries or less. With replacement $E(X) = 7 = 1/\mathcal{P}$ and it is possible you dont get in after $n$ tries for every integer $n$.

- **Question 8:** For $i \neq j$, $X_iX_j = 1$ only when both $i$ and $j$ get their own hats, an event with probability $\frac{(n-2)!}{n!}$. Therefore $E(X_iX_j) = \frac{1}{n(n-1)}$ and $\mathrm{cov}(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j) = 1/(n(n-1)) - 1/n^2 = 1/[n^2(n-1)]$. Though this is small, it is not zero, so the indicators cannot be independent.

- **Question 9:** This strongly resembles question 5 and is easily solved using indicators. Let $X_i$ be the indicator of the event that at least one of the people select floor $i$. $E(X_i) = 1 - (7/8)^5$. Writing $N = X_1 + \cdots + X_8$ as the number of different floors selected by the five people, $E(N) = 8 \cdot (1 - (7/8)^5) = 3.896728516$. When there are 10 floors $E(N) = 8 \cdot (1 - (7/8)^{10}) = 5.89539539$.

- **Question 11:** When $n$ balls are randomly placed in $n$ boxes, and $N_1$ denotes the number of boxes with exactly one ball, $N_1 = n$ only if each ball is in a different box, an event with probability $\frac{n!}{n^n}$.

  $N_1 = n - 1$ only if two balls are in the same box (say box $j$) and the other $n - 2$ balls are all in different boxes, none of which is box $j$. The probability of this event is

$$\frac{n\binom{n}{2}(n-1)_{n-2}}{n^n}$$

($n$ choices for $j$, $\binom{n}{2}$ choices for which balls go in box $j$, and $(n-1)_{n-2}$ ways to distribute the remaining balls).

Write $N_1 = X_1 + \cdots + X_n$, where $X_i$ is the indicator of the event that box $i$ has exactly one ball. Since $E(X_i) = n(n-1)^{n-1}/n^n = (1-1/n)^{n-1}$, $E(N_1) = n(1-1/n)^{n-1}$ a quantity that is $\sim n/e$ when $n$ is large.

- **Question 12:** When $n$ balls are randomly placed in $n$ boxes, $N_2$ (the number of boxes with exactly two balls) satisfies $N_2 = X_1 + \cdots + X_n$, where $X_i$ is the indicator of the event that box $i$ has exactly two balls. By simple counting

  $$E(X_i) = \frac{\binom{n}{2}(n-1)^{n-2}}{n^n} = \frac{1}{2}(\frac{n-1}{n})^{n-1}$$

  so $E(N_2) = (n/2)(1-1/n)^{n-1} \sim n/(2e)$ [$E(N_2) = (1/2)E(N_1)$ from Question 11:]

- **Question 13:** $S_{20}$ is the number of successes in 20 Bernoulli trials with $\mathcal{P} = 1/4$ as the prob. of success, so $E(S_{20}) = 5$ and $E(N) = 1/4$.