

The  $n$  by  $n$  **Hilbert matrix** is defined by

$$H_n = \begin{pmatrix} 1 & 1/2 & \dots & 1/n \\ 1/2 & 1/3 & \dots & 1/(n+1) \\ \cdot & \cdot & \dots & \cdot \\ 1/n & 1/(n+1) & \dots & 1/(2n-1) \end{pmatrix};$$

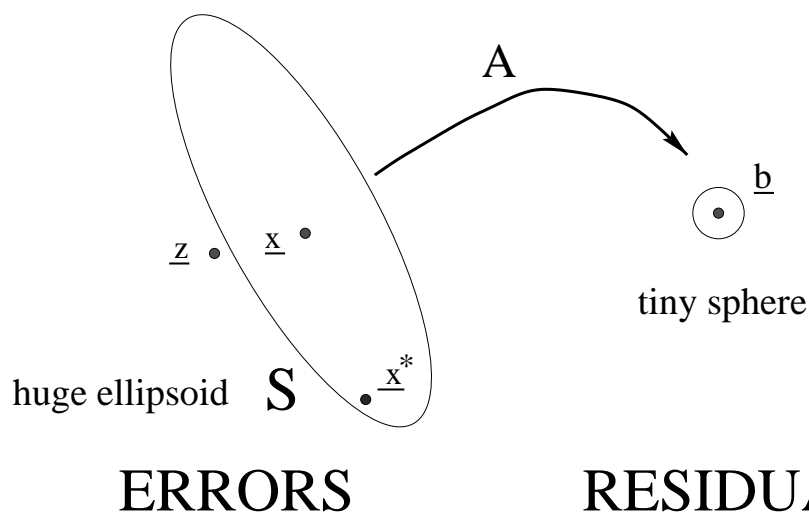
the row  $i$ , column  $j$  entry is  $h_{ij} = 1/(i+j-1)$ .

Consider the system  $A\underline{x} = \underline{b}$  where  $A = H_n$  and  $\underline{b}$  is the vector with  $b_i = \sum_{j=1}^n h_{ij}$ . Clearly this system has solution vector  $\underline{x}$  whose components are all ones.

The matrix  $A = H_n$  is notorious for propagating roundoff errors in Gaussian elimination. The following outputs are from my program to compute the *LUP* factorization of  $A$  using partial pivoting, and then, the forward-substitution (solution of  $L\underline{y} = \underline{b}(\underline{p})$ ) and backsolving  $U\underline{x} = \underline{y}$ . We denote the computed solution by  $\underline{x}^*$  and the **error** vector as  $\underline{e} \equiv \underline{x} - \underline{x}^*$ . Finally the **residual** vector is defined to be

$$\underline{r} = A\underline{e} = A(\underline{x} - \underline{x}^*) = \underline{b} - A\underline{x}^*.$$

The following pages of output show that **even though  $\underline{e}$  may be large (i.e., its length  $\|\underline{e}\| = \sqrt{e_1^2 + \dots + e_n^2}$  is large), the residual vector is small. This seems to be a property of the kind of errors committed by Gaussian elimination.**



The picture is meant to convey the idea that for certain matrices  $A$  (like  $H_n$ , e.g.) (i) the points  $\underline{z}$  that  $A$  maps “near” to  $\underline{b}$  can lie in a large set  $S$  (with points FAR from  $\underline{x} = A^{-1}\underline{b}$ ); (ii) Gaussian-Elimination computes a solution  $\underline{x}^* \in S$  (so it has small residual, even if it may have a large error); (iii) there are points  $\underline{z} \notin S$  that are close to  $\underline{x} = A^{-1}\underline{b}$  (small error) but still have large residuals (Gaussian elimination does not seem to find such points).

p	compact LU				L			U		
1	1.0000	0.5000	0.3333		1.0000	0.0000	0.0000	1.0000	0.5000	0.3333
2	0.5000	0.0833	0.0833		0.5000	1.0000	0.0000	0.0000	0.0833	0.0833
3	0.3333	1.0000	0.0056		0.3333	1.0000	1.0000	0.0000	0.0000	0.0056

y	x*	e	r
0.183333E+01	0.999999E+00	0.953674E-06	0.298023E-07
0.166667E+00	0.100001E+01	-0.548363E-05	0.298023E-07
0.555554E-02	0.999995E+00	0.536442E-05	0.447035E-07

|e| = 0.773024E-05

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p	compact LU				
1	1.0000	0.5000	0.3333	0.2500	0.2000
2	0.5000	0.0833	0.0833	0.0750	0.0667
5	0.2000	0.8000	0.0095	0.0150	0.0178
3	0.3333	1.0000	0.5833	-.0004	-.0008
4	0.2500	0.9000	0.8750	0.6428	0.0000

y	x*	e	r
0.228333E+01	0.100007E+01	-0.749826E-04	0.149012E-07
0.308333E+00	0.998680E+00	0.132036E-02	0.298023E-07
0.423016E-01	0.100548E+01	-0.548434E-02	-0.447035E-07
-0.126313E-02	0.991939E+00	0.806147E-02	-0.447035E-07
-0.113896E-04	0.100386E+01	-0.386322E-02	-0.149012E-07

|e|= 0.105707E-01

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p compact LU

1	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429
2	0.5000	0.0833	0.0833	0.0750	0.0667	0.0595	0.0536
7	0.1429	0.6429	0.0099	0.0161	0.0195	0.0213	0.0221
3	0.3333	1.0000	0.5600	-.0007	-.0014	-.0020	-.0024
5	0.2000	0.8000	0.9600	0.6429	0.0000	-.0001	-.0001
4	0.2500	0.9000	0.8400	0.9643	0.8439	0.0000	0.0000
6	0.1667	0.7143	1.0000	0.2976	0.6257	-.5555	0.0000

y	x*	e	r
0.259286E+01	0.100090E+01	-0.903249E-03	0.745058E-07
0.421429E+00	0.967215E+00	0.327848E-01	0.149012E-07
0.888072E-01	0.129571E+01	-0.295714E+00	0.447035E-07
-0.647800E-02	-0.945102E-01	0.109451E+01	0.447035E-07
-0.260744E-03	0.293098E+01	-0.193098E+01	0.745058E-07
0.351676E-05	-0.617661E+00	0.161766E+01	0.000000E+00
-0.165790E-06	0.151771E+01	-0.517712E+00	0.372529E-07

|e| = 0.281070E+01

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p compact LU

1	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111
2	0.5000	0.0833	0.0833	0.0750	0.0667	0.0595	0.0536	0.0486	0.0444
7	0.1429	0.6429	0.0099	0.0161	0.0195	0.0213	0.0221	0.0223	0.0222
3	0.3333	1.0000	0.5600	-.0007	-.0014	-.0020	-.0024	-.0028	-.0030
9	0.1111	0.5333	0.9503	-.4242	0.0000	0.0001	0.0002	0.0003	0.0004
4	0.2500	0.9000	0.8400	0.9643	-.6159	0.0000	0.0000	0.0000	0.0000
5	0.2000	0.8000	0.9600	0.6429	-.7298	0.8505	0.0000	0.0000	0.0000
6	0.1667	0.7143	1.0000	0.2976	-.4566	0.3754	0.1487	0.0000	0.0000
8	0.1250	0.5833	0.9800	-.2386	0.5076	-.1102	-.3953	-.3756	0.0000

y	x*	e	r
0.282897E+01	0.998807E+00	0.119287E-02	0.178814E-06
0.514484E+00	0.104947E+01	-0.494668E-01	-0.596046E-07
0.133351E+00	0.522337E+00	0.477663E+00	0.119209E-06
-0.122726E-01	0.258982E+01	-0.158982E+01	0.000000E+00
0.104370E-02	0.210264E+00	0.789736E+00	0.596046E-07
0.619291E-04	-0.551675E+01	0.651675E+01	-0.298023E-07
0.134884E-05	0.161007E+02	-0.151007E+02	0.596046E-07
0.257875E-06	-0.119670E+02	0.129670E+02	0.178814E-06
0.845820E-07	0.501332E+01	-0.401332E+01	0.894070E-07

|e| = 0.214040E+02

A similar phenomenon occurs in the computation of  $A^{-1}$ . When  $A$  is the Hilbert matrix we make large errors in the computation of  $(A^{-1})$ . Here is the true inverse of  $H_7$ , the 7 by 7 Hilbert matrix:  $H_7^{-1} =$

```
[ 49      -1176      8820      -29400      48510      -38808      12012]
[
[-1176     37632    -317520     1128960    -1940400     1596672    -504504]
[
[8820     -317520     2857680    -10584000    18711000    -15717240    5045040]
[
[-29400    1128960    -10584000    40320000    -72765000    62092800    -20180160]
[
[48510    -1940400    18711000    -72765000    133402500    -115259760    37837800]
[
[-38808    1596672    -15717240    62092800    -115259760    100590336    -33297264]
[
[12012    -504504    5045040    -20180160    37837800    -33297264    11099088]
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and here is what I computed by following the efficient algorithm using Gauss-Jordan reduction on the LUP factorization, answers rounded to the nearest integer ( $A^*$  is the computed inverse of  $H_7$ ).  $A^* =$

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  37.      -680.      3997.      -10499.      13635.      -8513.      2021.
 -682.     17701.    -123698.     369784.    -540135.     380627.    -103545.
 4031.    -124267.     979226.    -3228815.     5148214.    -3941198.     1162847.
-10673.    373653.    -3244812.    11590284.   -19798362.    16111001.    -5023300.
 14014.    -549598.     5200422.   -19887368.     35931044.   -30652272.     9951588.
 -8880.     390447.    -4002181.    16249666.   -30765518.     27254454.    -9128023.
 2152.    -107212.     1187148.    -5085478.     10019309.    -9154415.     3142867.
```

Though the relative errors in  $A^*$  are enormous, the product  $(H_7)(A^*)$  is not far from the identity matrix.