

(Write up convincing answers to the questions marked by an asterisk (*). Some of the unstarred questions will be worked in recitation. (**) indicates a more challenging problem, not required to hand in unless you want feedback - its not part of the grading. Your solutions are due in class on Monday, Feb. 6, 2012.

1. (Sets) Prove the following facts, in the manner indicated, in detail.
 - (a) (deMorgan 1) $(\bigcup_{i=1}^n A_i)^c = \bigcap_{i=1}^n (A_i^c)$. Do this both *by induction*, and *without* induction.
 - (b) (*) (from deMorgan 2) $(\bigcap_{i=1}^n (A_i \cup B_i))^c = \bigcup_{i=1}^n (A_i^c \cap B_i^c)$. Do this one by induction.
 - (c) (*) $B \cap (\bigcup_{i=1}^n A_i) = \bigcup_{i=1}^n (B \cap A_i)$. Do this *without* induction.
 - (d) $B \cup (\bigcap_{i=1}^n A_i) = \bigcap_{i=1}^n (B \cup A_i)$. Do this both *by induction* and *without* induction.

2. (Sets) Decide whether the following statements are TRUE or FALSE and give a convincing argument to support your claim; if FALSE, you should offer a concrete counterexample.
 - (a) (*) $(A \cup B) \setminus C = A \cup (B \setminus C)$.
 - (b) $(A \cup B) \setminus A = B$.
 - (c) (*) $(A \setminus B) \cup B = A$
 - (d) $(A \cup B) = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$.
 - (e) (*) $(A \cap B)^c \setminus A^c = B^c$
 - (f) $\bigcap_{i=1}^n A_i \subseteq A_1$.
 - (g) (*) $A_1 \subset \bigcup_{i=1}^n A_i$.
 - (h) $\bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^{m+n} A_i$, $m > 0$; (i.e., $\bigcup_{i=1}^k A_i$ increases with k).
 - (i) (*) $\bigcap_{i=1}^n A_i \subseteq \bigcap_{i=1}^{m+n} A_i$, $m > 0$; (i.e., $\bigcap_{i=1}^k A_i$ increases with k).
 - (j) (**) $A \cap B = (A \cap B \cap C) \cup (A \cap B \cap D)$.

3. (Random Experiments and Sample Spaces) For each of the following random experiments, carefully describe the sample space, S . Try to compute $|S|$, the size of S , and explain your reasoning.
 - (a) (*) A coin is tossed 3 times.
 - (b) A die is thrown 3 times.
 - (c) (*) Three people enter the elevator in the basement of a building with 6 floors. Each states where he will get out.
 - (d) A box has 10 chips. The first is numbered with a "1", the second with a "2", etc. A chip is drawn from the box, its value noted, and then it is returned to the box. Then a second chip is drawn and its value noted.
 - (e) (*) As in (d) but the chips are NOT returned to the box.
 - (f) (*) A die is tossed. If it shows an EVEN face, a coin is thrown. Otherwise (the die showed an ODD face), the die is thrown 2 *more* times and the results are written down.
 - (g) A 4 node, rooted binary tree is written down.
 - (h) (*) Toss a coin. If it comes up HEAD, do the hat experiment with $n = 2$ hats. Otherwise do the hat experiment with $n = 3$ hats.
 - (i) (**) The hatcheck experiment with $n = 4$ hats. How many outcomes are derangements (nobody gets their own hat)? Any conjecture about the behaviour of the fraction of derangements in a random permutation of $1, \dots, n$?

4. (Events) Carefully describe the events A , B , $A \cup B$, and $A \cap B$ in the following sample spaces from 3, above, and determine the sizes of these events.
- (a) In 3a, $A = \{\text{Head on the first and last tosses}\}$, $B = \{\text{at least 2 tails}\}$.
 - (b) (*) In 3b, $A = \{\text{at least one even-score face}\}$, $B = \{\text{all faces 4 or more}\}$.
 - (c) (*) In 3d, $A = \{\text{same chip both times}\}$, $B = \{\text{chip 10 is not chosen}\}$.
 - (d) In 3f, $A = \{\text{the coin is NOT thrown}\}$, $B = \{\text{exactly one dice throw showed a four}\}$.
 - (e) In 3g, $A = \{\text{the root has two children}\}$, $B = \{\text{the tree has depth 4}\}$.
 - (f) In 3h $A = \{\text{person 2 gets her own hat}\}$, $B = \{\text{person 1 does not get her own hat}\}$.
 - (g) (**) As in 3d, a box has 3 red and 5 black chips. In this experiment, you pick a chip, note its color, and return it to the box. The experiment continues *until* you have picked more red than black chips. (i) Describe the sample space and then let $A = \{\text{stop on third pick}\}$, $B = \{\text{stop before sixth pick}\}$.