CS 510 Homework - due December 7, 2011

Turn in only the first five for grading

I. How many subintervals would you use to approximate $\int_0^{\pi/2} x \sin x \, dx$ to within 10^{-8} via (i) the trapezoidal rule? (ii) Simpson's rule?

II. Simpson's rule, $\int_a^{a+2h} f(x)\,dx \approx \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)]$, is obtained by integrating $p_2(x)$, the quadratic interpolating polynomial based on the values $f(a), f(a+h), f(a+2h)$. The error term is $\int_a^{a+2h} (x-a)(x-(a+h))(x-(a+2h)) \frac{f^{(3)}(\xi)}{3!}\,dx$. Note that the polynomial part of this integral changes sign over the interval $[a, a+2h]$; thus one <u>cannot</u> take $f^{(3)}(\xi)$ outside the integral to simplify the error term.

Now consider approximating $\int_a^{a+2h} f(x)\,dx$ by $\int_a^{a+2h} p_3(x)\,dx$ where $p_3(x)$ is a cubic interpolating polynomial based on the values $f(a), f(a+h), f(a+2h)$, and $f'(a+h)$. One may write $p_3(x) = p_2(x) + c(x)$, where the "correction" $c(x)$ has the form $c(x) = \text{scalar} \cdot (x-a)(x-(a+h))(x-(a+2h))$. Note that $\int_a^{a+2h} c(x)\,dx = 0$, so $\int_a^{a+2h} p_3(x)\,dx$ also yields Simpson's rule. Use this second interpretation of Simpson's rule to obtain the error formula

$$\int_a^{a+2h} f(x)\,dx - \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)] = -\frac{h^5}{90} f^{(4)}(\xi), \;\; \xi \in (a, a+2h).$$

Extend this to obtain the error formula for the composite version of Simpson's rule.

III. Determine the weights and error term for the quadrature formula

$$\int_a^{a+h} f(x)\,dx \approx w_0 f(a) + w_1 f(a-h) + w_2 f(a-2h),$$

based on quadratic interpolation of $f(x)$ at $x = a, a-h, a-2h$. (This *uncentered* formula is potentially useful for solving ordinary differential equations.)

IV. If the integrand in $\int_{-1}^{1} f(x)\,dx$ is approximated by a cubic Hermite interpolating polynomial based on the values of $f(x)$ and $f'(x)$ at $x = \pm\alpha$, the resulting quadrature formula will have the form

$$\int_{-1}^{1} f(x)\,dx \approx w_0 f(\alpha) + w_1 f(-\alpha) + w_2 f'(\alpha) + w_3 f'(-\alpha). \qquad (*)$$

(i) Determine the weights by enforcing exactness for $f(x) = 1, x, x^2, x^3$.
(ii) Derive the error term.
(iii) For what value of α will w_2 and w_3 be zero? Your error term for $(*)$ should have the form $c(\alpha)h^4 f^{(4)}(\xi)$. What value of α minimizes $c(\alpha)$?

V. Approximate $\int_0^{2\pi/3} \sin x\,dx$ by Romberg integration using $1, 2, 4$ subintervals.

VI. Let $p_n(x)$ be the interpolating polynomial for $f(x)$ based on samples at $x_0, ..., x_n$. The approximation

$$f'(a) \approx p_n'(a)$$

leads to a numerical differentiation formula of the form

$$f'(a) \approx \sum_{i=0}^{n} w_i f(x_i)$$

whose weights w_i can be determined by imposing the appropriate exactness conditions. The error term for this formula is

$$f'(a) - p_n'(a) = \frac{d}{dx}\left[(x-x_0)\cdots(x-x_n)\frac{f^{(n+1)}(\xi)}{(n+1)!}\right]\Bigg|_{x=a}$$

where ξ depends on x. In the case where the evaluation point is one of the sample points, $a = x_k$, say, the error term is easy to simplify. Writing

$$f'(x_k) - p_n'(x_k) = \frac{d}{dx}\left[u(x)v(x)\right]\Bigg|_{x=x_k}$$

where

$$u(x) = x - x_k, \quad v(x) = \Pi_{j\neq k}(x-x_j)\frac{f^{(n+1)}(\xi)}{(n+1)!},$$

we have

$$\begin{aligned} f'(x_k) - p_n'(x_k) &= u(x_k)v'(x_k) + u'(x_k)v(x_k) = v(x_k) \\ &= \Pi_{j\neq k}(x_k - x_j)\frac{f^{(n+1)}(\xi)}{(n+1)!}, \quad \xi \in (\min\{x_0, ..., x_n\}, \max\{x_0, ..., x_n\}). \end{aligned}$$

Derive a numerical differentiation formula of the form

$$f'(a) \approx w_0 f(a) + w_1 f(a+h) + w_2 f(a+2h)$$

based on quadratic interpolation of $f(x)$. What is the error term for the formula?

VII.

(i) Write the error term in the numerical differentiation formula

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

as a power series in h. [Hint: Expand $f(a+h)$ in a Taylor series about a.]

(ii) Given the function values (taken from $f(x) = e^x + x$):

x	$f(x)$
0.0	1.00000
0.1	1.20517
0.2	1.42140
0.4	1.89182
0.8	3.02554

approximate $f'(0)$ by applying the formula in (i) with $h = .8, .4, .2, .1$, and performing repeated extrapolation.

VIII. Derive a Gaussian quadrature formula of the form

$$\int_{-1}^{1} f(x)\,dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

What is its error term?