CS 510 Homework - due November 29, 2011

I. (a) Approximate $\ln 1.5$ via interpolation using the values $\ln 1 = .0000$, $\ln 2 = .6931$, $\ln 3 = 1.0986$. Bound the error in your approximation using the error term for the interpolating polynomial.

(b) Same as (a), except add the derivative of $\ln x$ at $x = 1$ to the data on which the interpolating polynomial is based. Note: The divided difference table and interpolating polynomial for part (a) can be extended into those for part (b).

II. Suppose $f(x) = \ln x$ is to be approximated over $x \in [1, 3]$ using equally spaced points

$$1 = x_0 < x_1 < \cdots < x_n = 3.$$

How large must n be to achieve an absolute error $\leq 10^{-8}$ using the following interpolation schemes:

(i) Approximation by linear interpolation in each subinterval.

(ii) Approximation by quadratic interpolation over each pair of subintervals. (You may use the fact that $\max_{a \leq x \leq a+2h} |(x-a)(x-(a+h))(x-(a+2h)| = \frac{2\sqrt{3}}{9}h^3$.)

(iii) Cubic Hermite interpolation over each subinterval.

(iv) Approximation by an n^{th} degree interpolating polynomial. (You may use the fact that $\max_{[x_0,x_n]} |(x-x_0)\cdots(x-x_n)| < n!\,h^{n+1}$ for grid points $\{x_0, \ldots, x_n\}$ with uniform spacing h).

III. Given sample points $(x_0, f_0), \ldots, (x_n, f_n)$, let $p_{i,\ldots,j}(x)$ denote the interpolating polynomial based on $(x_i, f_i), (x_{i+1}, f_{i+1}), \ldots, (x_j, f_j)$, $j \geq i$.

(i) Show that for $j > i$

$$p_{i,\ldots,j}(x) = \frac{(x - x_i)p_{i+1,\ldots,j}(x) - (x - x_j)p_{i,\ldots,j-1}(x)}{x_j - x_i}. \qquad (1)$$

[Hint: Write $p_{i,\ldots,j}(x)$ as $p_{i,\ldots,j-1}(x) + c(x)$ and as $p_{i+1,\ldots,j}(x) + d(x)$, then take an appropriate combination of the two.]

(ii) To evaluate $p_{0,\ldots,n}(x)$ at a point $x = z$, one may use a structure analogous to the divided difference table. The entries in the table are now $p_{i,\cdots,j}(z)$, computed via (1), rather than $[x_i, \ldots, x_j]$. Use this interpolation scheme to approximate $\ln 1.5$ using the sample values $\ln 1 = .0000$, $\ln 2 = .6931$, $\ln 3 = 1.0986$. Note: a potential advantage of this interpolation scheme is that one gets *many* approximations to the desired quantity, which can be used as a basis for empirical error estimation.

IV. Let $p_k(x)$ denote the interpolating polynomial based on $f(x_0), \ldots, f(x_k)$ where $x_0, \ldots, x_k$ are distinct from one another. The basis of our construction of the Newton form of the interpolating polynomial was the expression

$$p_k(x) = p_{k-1}(x) + (x - x_0)\cdots(x - x_{k-1})\,[x_0, \ldots, x_k].$$

Taking $x = x_k$, we have

$$f(x_k) = p_k(x_k) = p_{k-1}(x_k) + (x_k - x_0)\cdots(x_k - x_{k-1})\,[x_0, \ldots, x_k].$$

On the other hand, the error formula for the interpolating polynomial gives

$$f(x_k) = p_{k-1}(x_k) + (x_k - x_0)\cdots(x_k - x_{k-1})\frac{f^{(k)}(\xi)}{k!}$$

for some $\xi \in (\min\{x_0, \ldots, x_k\}, \max\{x_0, \ldots, x_k\})$. Comparing, we conclude that

$$[x_0, \ldots, x_k] = \frac{f^{(k)}(\xi)}{k!}$$

for some $\xi \in (\min\{x_0, \ldots, x_k\}, \max\{x_0, \ldots, x_k\})$.

(i) Consider the divided difference table for constructing $p_k(x)$ in Newton form using samples $\{(x_i, f(x_i)),\ i = 0, \ldots, k\}$ What limits do the coefficients appearing in the table approach as all x_i's approach a common limit, $x = x_0$, say? What do $p_k(x)$ and its error term become in this limit?

(ii) Approximate $\ln(1.5)$ via polynomial interpolation using the following values for $f(x) = \ln x$: $f(1), f'(1), f''(1), f(2)$. Bound the error in your approximation using the error formula for the interpolating polynomial.

V. (i) In class, we derived a system of equations that can be used to determine a cubic spline interpolant $S(x)$ for $f(x_0), ..., f(x_n)$ subject to prescribed endpoint derivatives $f'(x_0), f'(x_n)$. Obtain a modified system for the case where the given $f'(x_0)$ condition is replaced by:

(a) a *natural boundary condition* at x_0 (i.e., $S''(x_0) = 0$).

(b) a *not-a-knot* condition (i.e., third derivative continuity) at $x = x_1$.

(ii) We wish to determine a cubic spline interpolant $S(x)$ for $f(x) = \sin \frac{\pi}{2} x$ over $[0, 3]$ based on the values $f(0), f(1), f(2), f(3), f'(3)$ and a not-a-knot condition at $x = 1$. What is $S'(2)$? [Note: You should be able to set this up as a single equation for $S'(2)$.]

VI. Determine a cubic spline $B(x)$ based on nodes at $..., -3, -2, -1, 0, 1, 2, 3, ...$ which satisfies:

$$\begin{aligned} B(x) &\equiv 0 \quad \text{for } x \notin [-2, 2], \\ B(0) &= 1. \end{aligned}$$

[Hint: $B(x)$ must be symmetric about $x = 0$; therefore, $B'(0) = 0$. Setting $B(1) = \alpha$, one can generate $B(x)$ in $[1, 2]$, then impose the appropriate continuity conditions at $x = 1$ to determine the correct value of α.] $B(x)$ is the canonical *B-spline* basis function for representing cubic splines. Note that it is nonzero over 4 subintervals - this is the minimum number for a cubic spline that is not identically zero.