Solutions to additional problems

I.
(ii) Take $\sigma_i = |\lambda_i|$. If $\lambda_i > 0$, take column i of $\{U, V\} = \{v_i, v_i\}$ or $\{-v_i, -v_i\}$. If $\lambda_i < 0$, take column i of $\{U, V\} = \{v_i, -v_i\}$ or $\{-v_i, v_i\}$.

II.
(i) $\|A\|_2 = \|U\Sigma V_T\|_2 = \|\Sigma\|_2 = \sigma_1$
(ii) $A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$
(iii) $\kappa_2(A) = \|A\|_2\|A\|_2 = \frac{\sigma_1}{\sigma_n}$.
(iv) $\|\delta x\|_2 = \|A^{-1}\delta b\|_2 = \|A^{-1}\|_2\|\delta b\|_2 \Longrightarrow \delta b =$ multiple of u_n (last column of U).
$\|b\|_2 = \|Ax\|_2 = \|A\|_2\|x\|_2 \Longrightarrow x =$ multiple of v_1 (first column of V) $\Longrightarrow b =$ multiple of u_1 (first column of U).

III.

$$A_{m\times n}x = b \Rightarrow U\Sigma V^T = b \Rightarrow \Sigma \underbrace{V^T x}_{\xi} = \underbrace{U^T b}_{\beta}$$

$$\left[\begin{array}{ccc|ccc} \sigma_1 & & & 0 & \dots & 0 \\ & \ddots & & & & \\ & & \sigma_m & 0 & \dots & 0 \end{array}\right] \left[\begin{array}{c} \xi_1 \\ \vdots \\ \xi_m \\ \hline \xi_{m+1} \\ \vdots \\ \xi_n \end{array}\right] = \left[\begin{array}{c} \beta_1 \\ \vdots \\ \beta_m \end{array}\right]$$

Solution: $\xi_i = \frac{\beta_i}{\sigma_i},\ i = 1, ..., m;\quad \xi_{m+1}, ..., \xi_n$ arbitrary.
Note that $\|x\|_2 = \|\xi\|_2$, so mininum norm solution gotten by taking $\xi_{m+1} = \cdots = \xi_n = 0$. Then form $x = V\xi$.

IV. (i) overdetermined system $Ac = y$:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$$

solution via normal equations $A^T Ac = A^T y$...

$$\left[\begin{array}{cc|c} 5 & 10 & 11 \\ 10 & 30 & 31 \end{array}\right] \Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} .4 \\ .9 \end{bmatrix}$$

(ii) Write approximation as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^* \\ y^* \end{pmatrix} + \xi \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad q_1^2 + q_2^2 = 1,$$

where x^*, y^*, q_1, q_2 are to be determined.

Here

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} 2 \\ 2.2 \end{pmatrix}; \qquad M = \begin{pmatrix} -2 & -1.2 \\ -1 & -1.2 \\ 0 & -1.2 \\ 1 & 1.8 \\ 2 & 1.8 \end{pmatrix}.$$

SVD (via Matlab):

$$\begin{aligned} M &= U\Sigma V^T \\ &= \begin{pmatrix} -0.5106 & 0.5221 & 0.3970 & -0.0200 & 0.5555 \\ -0.3537 & -0.0906 & -0.6534 & 0.6146 & 0.2491 \\ -0.1968 & -0.7033 & 0.5745 & 0.3604 & 0.0824 \\ 0.4522 & 0.4422 & 0.2835 & 0.6987 & -0.1774 \\ 0.6091 & -0.1705 & -0.0714 & -0.0621 & 0.7688 \end{pmatrix} \begin{pmatrix} 4.4056 & 0 \\ 0 & 1.1795 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.6912 & -0.7226 \\ 0.7226 & 0.6912 \end{pmatrix}^T \end{aligned}$$

Straight line approximation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2.2 \end{pmatrix} + \xi \begin{pmatrix} .6912 \\ .7226 \end{pmatrix}, \qquad \sqrt{\sum_{i=1}^{5} d_i^2} = 1.1795$$

Equivalently,

$$y = .1091 + 1.0454x$$