Additional problems

I. (i) Using the fact that the singular values of A are the square roots of the eigenvalues of A^TA and the columns of U are a corresponding set of orthonormal eigenvectors for A^TA, determine an SVD for the matrix

$$\begin{bmatrix} -3 & 0 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

Check via Matlab (type *'Matlab'*, then *'help svd'* to see how to do this).

(ii) Let $A_{n\times n}$ be a real symmetric matrix with eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ and corresponding orthonormal eigenvectors $\{v_1, v_2, ..., v_n\}$. Determine an SVD for A.

II. Let $A_{n\times n}$ have SVD $U\Sigma V^T$ with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$.
(i) Show that $\|A\|_2 = \sigma_1$
(ii) What is the SVD of A^{-1}?
(iii) Show that $\kappa_2(A) = \frac{\sigma_1}{\sigma_n}$.
(iv) Given two systems $Ax = b$ and $A\,\delta x = \delta\, b$, for what vectors b and δb is equality achieved in the 2-norm condition number bound, i.e., $\frac{\|\delta x\|_2}{\|x\|_2} = \|A^{-1}\|_2 \|A\|_2 \frac{\|\delta b\|_2}{\|b\|_2}$.

III. Consider an underdetermined system

$$A_{m\times n}x = b, \qquad m < n$$

where A has maximum rank (i.e., $\text{rank}(A) = m$). Such a system will have infinitely many solutions. Describe how an SVD for A can be used to determine the solution x which minimizes $\|x\|_2$.

IV. We wish to approximate the points $(0,1), (1,1), (2,1), (3,4), (4,4)$ by a straight line of the form $y(x) = c_1 + c_2 x$.
(i) Find the straight line which minimizes $\sum_i d_i^2$ where d_i is the vertical distance from (x_i, y_i) to the approximating straight line.
(ii) Find the straight line which minimizes $\sum_i \delta_i^2$ where δ_i is the orthogonal distance from (x_i, y_i) to the approximating straight line.