CS 510 Homework - solutions to extra problems

I.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(i) <u>Jacobi</u>
component form:

$$x_1^{(k+1)} = 1 + \frac{1}{2} x_2^{(k)}, \ x_2^{(k+1)} = \frac{1}{2} + \frac{1}{2} x_1^{(k)}$$

matrix form:

$$x^{(k+1)} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} x^{(k)} + \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x^{(1)} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

<u>Gauss-Seidel</u>
component form:

$$x_1^{(k+1)} = 1 + \frac{1}{2} x_2^{(k)}, \ x_2^{(k+1)} = \frac{1}{2} + \frac{1}{2} x_1^{(k+1)}$$

matrix form:

$$x^{(k+1)} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix} x^{(k)} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>SOR</u>
component form:

$$x_1^{(k+1)} = x_1^{(k)} + \omega \left[(1 + \frac{1}{2} x_2^{(k)}) - x_1^{(k)} \right],$$
$$x_2^{(k+1)} = x_2^{(k)} + \omega \left[(\frac{1}{2} + \frac{1}{2} x_1^{(k+1)}) - x_2^{(k)} \right]$$

matrix form:

$$x^{(k+1)} = \begin{bmatrix} 1-\omega & \omega/2 \\ (1-\omega)\omega/2 & 1-\omega+\omega^2/4 \end{bmatrix} x^{(k)} + \begin{bmatrix} \omega \\ \omega/2 + \omega^2/2 \end{bmatrix}$$

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x^{(1)} = \begin{pmatrix} \omega \\ \omega/2 + \omega^2/2 \end{pmatrix}$$

(ii) Number of Jacobi iterations required to reduce initial error by factor 10^{-6}...
For Jacobi $\|G\|_\infty = 1/2$.

$$\|e^{(k)}\| / \|e^{(0)}\| \le \|G\|^k = (1/2)^k \le 10^{-6} \implies k \ge 20$$

II. Iteration for $Ax = b$:

$$Mx^{(k+1)} = -Nx^{(k)} + b, \quad \text{where } A = M + N.$$

Jacobi: $M = D,\ N = A - D$.
Gauss-Seidel: $M = D + L,\ N = U$.
SOR: $M = \omega^{-1}D + L,\ N = U + (1 - \omega^{-1})D$.

III.

$$x^{(k+1)} = x^{(k)} - \alpha\left(Ax^{(k)} - b\right), \quad \alpha \text{ a scalar.}$$

(i) iteration matrix G ...

$$x^{(k+1)} = (I - \alpha A)x^{(k)} - \alpha b \qquad \Longrightarrow G = I - \alpha A.$$

(ii) Assume A strictly diagonally dominant with rows scaled so that $a_{i,i} = 1$ for all i. Determine value of α for which $\|G\|_\infty$ is minimized ...

Since $a_{i,i} = 1$

$$G = \begin{pmatrix} 1-\alpha & -\alpha a_{1,2} & \cdots & -\alpha a_{1,n} \\ -\alpha a_{2,1} & 1-\alpha & \cdots & -\alpha a_{2,n} \\ & \vdots & & \\ -\alpha a_{n,1} & -\alpha a_{n,2} & \cdots & 1-\alpha \end{pmatrix}.$$

$$\|G\|_\infty = \max_i \sum_j |G_{i,j}| \ = |1-\alpha| + |\alpha| \sum_{j \neq i} |a_{ij}|.$$

Minimum is achieved for $\alpha = 1$, in which case $\|G\|_\infty = \max_{j \neq i} |a_{i,j}| < 1$.

IV. Gaussian elimination applied to tridiagonal system

$$(1) \qquad \begin{bmatrix} d_1 & u_1 & & \\ l_2 & d_2 & u_2 & \\ & & \ddots & \\ & & l_n & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Odd-even reduction procedure; assume $n = 2^p - 1$ ...
Eliminate x_{i-1} and x_{i+1} from equation i using equations $i-1$ and $i+1$. Result is tridiagonal system of size $2^{p-1} - 1$ for even x_i's:

$$(2) \qquad \begin{bmatrix} d'_2 & u'_2 & & \\ l'_4 & d'_4 & u'_4 & \\ & & \ddots & \\ & & l'_{n-1} & d'_{n-1} \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b'_2 \\ b'_4 \\ \vdots \\ b'_{n-1} \end{bmatrix}$$

Once solution of reduced system (2) is known, use odd-indexed equations in (1) to 'backsolve' for odd x_i's. Apply reduction/backsolve procedure recursively until a 1 by 1 system results.

Solve following tridiagonal system via this procedure:

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

First reduction gives:

$$\begin{bmatrix} 1 & -1/2 & \\ -1/2 & 1 & -1/2 \\ & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$$

Next reduction gives:

$$1/4 \; x_4 = 1/4$$

"Backsolving" ...

$$x_4 = 1$$

$$x_2 = 1$$
$$x_6 = 1$$

$$x_1 = 1$$
$$x_3 = 1$$
$$x_5 = 1$$
$$x_7 = 1$$

Operation count for system of size $n = 2^p - 1$...

Let $S(p)$ = number of operations to solve tridiagonal system of size $n = 2^p - 1$.

$$\begin{aligned} S(p) = &\text{ no. of ops to eliminate odd } x_i\text{'s from even eqns} \\ &+ \text{no. of ops to solve tridiag system of size } n = 2^{p-1} - 1 \\ &+ \text{no. of ops to solve for odd } x_i\text{'s once even ones are known} \end{aligned}$$

$$
\begin{aligned}
S(p) &= 8 \cdot 2^{p-1} + S(p-1) \, + \, 3 \cdot 2^{p-1} \\
&= 11 \cdot 2^{p-1} \, + \, S(p-1)
\end{aligned}
$$

$$
\begin{aligned}
S(p) &= 11 \cdot (2^{p-1} + 2^{p-2}) \, + \, S(p-2) \\
&= 11 \cdot (2^{p-1} + 2^{p-2} + 2^{p-3}) \, + \, S(p-3) \\
&\vdots \\
&= 11 \cdot (2^{p-1} + \cdots + 2^{p-k}) \, + \, S(p-k)
\end{aligned}
$$

Take $k = p - 1$ to get:

$$
\begin{aligned}
S(p) &= 11 \cdot (2^{p-1} \, + \, \cdots + 2) + \, S(1) \\
&= 11 \cdot 2^p \, (1/2 + 1/4 + \cdots) \\
&\cong 11n
\end{aligned}
$$

In all, there are $O(\log n)$ eliminate and backsolve steps, each of which can be done in parallel.