

CS 510 Homework - due October 11, 2011

I. Consider the linear algebraic system

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix}.$$

(i) Using Gaussian elimination with partial pivoting, obtain a $PA = LU$ decomposition of the coefficient matrix.

(ii) Use the results of (i) to solve the given system. (Feel free to use Matlab function `'lu'` to check your answers to (i) and (ii) - type `'help lu'` for information on how to use.)

(iii) Use the results of (i) to invert the coefficient matrix. (You can check your answer via Matlab function `'inv'`.)

II. Obtain a Cholesky ($\tilde{L}\tilde{L}^T$) decomposition of

$$\begin{bmatrix} 4 & -2 & & \\ -2 & 2 & -1 & \\ & -1 & 2 & 2 \\ & & 2 & 5 \end{bmatrix}.$$

(You can check your answer via Matlab function `'chol'`.)

III. Let $A^{(1)} = A$ and $A^{(k)}$, $k = 2, \dots, n$, the result of applying Gaussian elimination up through column $k - 1$ of a nonsingular $n \times n$ matrix A (thus $A^{(n)} = U$). From a numerical stability standpoint, the objective of pivoting is to avoid excessive growth in

$$\rho(A) = \frac{\max_{i,j,k} |A_{i,j}^{(k)}|}{\max_{i,j} |A_{i,j}^{(1)}|}.$$

(i) Show that if partial pivoting is used, then

$$\frac{\max_{i,j} |A_{i,j}^{(k+1)}|}{\max_{i,j} |A_{i,j}^{(k)}|} \leq 2.$$

Note that this gives a bound $\rho(A) \leq 2^{n-1}$ for partial pivoting over all nonsingular matrices A .

(ii) Let M_n denote the $n \times n$ matrix with entries m_{ij} defined by

$$m_{ij} = \begin{cases} 1, & i = j \text{ or } j = n, \\ -1, & i > j, \\ 0, & \text{otherwise.} \end{cases}$$

Determine $\rho(M_5)$ assuming partial pivoting is used. What is $\rho(M_n)$? Note: For matrices A arising in real world applications, it is exceedingly rare for $\rho(A)$ to exceed 10 for partial pivoting, making it the default pivoting scheme of choice.

(iii) For $A = M_4$ as defined in (ii), obtain a $PAQ = LU$ factorization using complete pivoting. What is $\rho(M_4)$ for this pivoting scheme?

(iv) Use the decomposition of (iii) to solve $M_4 x = (2, 3, 0, 2)^T$.

IV. As a prelude to this question, note that matrix multiplication can be viewed in terms of submatrices. For example, if $A_{m \times n}$ and $B_{n \times p}$ are partitioned as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where, for $i = 1, 2$ and $j = 1, 2$, A_{ij} is $m_i \times n_j$ and B_{ij} is $n_i \times p_j$, then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

(i) Assume A is a nonsingular $n \times n$ matrix whose triangular factors $LU = A$ are known, and that we wish to compute the analogous factors \mathcal{L}, \mathcal{U} of an $(n+1) \times (n+1)$ matrix of the form

$$B = \begin{pmatrix} A & \alpha \\ \beta^T & \gamma \end{pmatrix}$$

where α and β are n -vectors, and γ is a scalar. Describe how L and U can be used for this purpose. How many additional operations are needed? [Hint: Write \mathcal{L}, \mathcal{U} in the form

$$\mathcal{L} = \begin{pmatrix} L & 0 \\ \lambda^T & 1 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} U & \mu \\ 0^T & \nu \end{pmatrix}.]$$

(ii) Let A_k denote the k -th principal submatrix of an $n \times n$ matrix A , i.e.,

$$A_k = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}.$$

One way to compute the LU factorization of A would be to start with that of A_1 and compute, in succession, those of A_2, \dots, A_n via the procedure in part (i). How many operations are needed to compute L and U in this way?

V. Show that the 1-norm of a matrix $A_{n \times n}$ is given by

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{i,j}|.$$

For what vector $x \neq 0$ is $\|Ax\|_1 = \|A\|_1 \|x\|_1$?

VI. For the matrix A in problem I:

(i) Find $\|A\|_1$, $\|A^{-1}\|_1$ and $\kappa_1(A)$.

(ii) Suppose \hat{x} is an approximate solution to $Ax = b$ where $\|b\|_1 = 1$. Given that $\|A\hat{x} - b\|_1 \leq .01$, bound the absolute and relative error in \hat{x} ($\|\hat{x} - x\|_1$ and $\|\hat{x} - x\|_1 / \|x\|_1$).

VII. For the matrix A in problem I, find righthand side vectors b and $b + \delta b$ such that equality is achieved in the condition number bound as applied to $Ax = b$ and $A(x + \delta x) = b + \delta b$:

$$\frac{\|\delta x\|_1}{\|x\|_1} = \kappa_1(A) \frac{\|\delta b\|_1}{\|b\|_1}.$$

VIII. Suppose A is nonsingular and consider the pair of linear systems:

$$Ax = b, \quad (A + \delta A)(x + \delta x) = b.$$

Show that

$$\frac{\|\delta x\|}{\|x + \delta x\|} \leq \kappa(A) \frac{\|\delta A\|}{\|A\|}.$$

Thus $\kappa(A)$ measures the sensitivity of the solution to $Ax = b$ with respect to changes in A as well as in b .

IX. (i) Suppose A is a nonsingular matrix and u, v are vectors (in which case $u^T v$ is a scalar and uv^T is a matrix). Assuming $v^T A^{-1} u \neq 1$, verify that

$$(A - uv^T)^{-1} = A^{-1} + cA^{-1}uv^T A^{-1}$$

where

$$c = \frac{1}{1 - v^T A^{-1} u}.$$

(ii) Assume that for a linear system with $A_{n \times n}$ as coefficient matrix, it takes $\alpha(n)$ operations to decompose A into its LU factors and $\beta(n)$ operations to perform the triangular solves. How many operations will it take to solve

$$(A - uv^T)_{n \times n} x = b$$

using the above formula? No need to invert any matrices!

(iii) Suppose we've computed a solution of $Ax = b$ via Gaussian elimination with partial pivoting, using 'decomp' and 'solve' routines taking $\frac{1}{3}n^3$, n^2 operations, respectively. We then learn that a single element, $A_{k,l}$, say, has been incorrectly specified. Explain how the above formula can be used to efficiently compute the solution of 'correct' system. How many operations are required?

X. (i) Show that

$$\kappa(A) = \max_{\|x\|=1} \|Ax\| / \min_{\|x\|=1} \|Ax\|.$$

(ii) Let $Q_{n \times n}$ be an orthogonal matrix:

- (a) Show that $\|Q\|_2 = 1$.
- (b) What is $\kappa_2(Q)$?