CS 510 Homework - due 9/27/11

I. (i) Find the Taylor polynomial which matches $f(x) = \ln x$ and its first three derivatives at $x = 1$.
(ii) Suppose the polynomial in (i) is used to approximate $\ln 1.1$. What is the resulting approximation? Bound its relative error using the error formula for Taylor polynomials.

II. (i) Show that the error in the numerical differentiation formula

$$f'(a) \approx D(h) \equiv \frac{f(a+h) - f(a)}{h}$$

can be expressed in the form $h/2 f''(\xi), \ \xi \in (a, a+h)$. [Hint: Write $f(a+h)$ as a linear Taylor expansion + remainder term about $x = a$.]
(ii) Suppose $D(h)$ is evaluated on a computer with unit roundoff error ϵ; i.e., for any nonzero real number x, the corresponding floating point approximation $\hat{x}$ satisfies $|(\hat{x} - x)/x| < \epsilon$, or, equivalently,

$$(*) \qquad \hat{x} = x(1 + \delta), \ \text{where } |\delta| < \epsilon.$$

Assuming the only roundoff error incurred is the approximation of $f(a)$ and $f(a+h)$ by $\widehat{f(a)}$ and $\widehat{f(a+h)}$ in accordance with $(*)$ (a reasonable assumption), show that the maximum error in the computed numerical derivative is bounded by

$$E(h) = \frac{2\epsilon M_0}{h} + \frac{h}{2} M_2$$

where M_0 and M_2 are bounds on $f(x)$ and $f''(x)$ in the vicinity of $x = a$. What h minimizes $E(h)$? (It will depend on ϵ, M_0, and M_2).

(iii) Suppose $f(x) = x^2$, $a = 1.3$, and the machine is an IEEE double precision machine. Estimate the value of h which for which $D(h)$ gives the best accuracy. Check your result via a simple MATLAB program that computes $D(h)$ for $h = 10^{-k}, \ k = 0, 1, ..., 20$.

III. Assume we wish to compute $\sqrt{a}$ where a is positive. Following are three fixed point equations with solution $\overline{x} = \sqrt{a}$, $a > 0$:

(a) $x = g_1(x) = a + x - x^2$

(b) $x = g_2(x) = 1 + x - \dfrac{x^2}{a}$

(c) $x = g_3(x) = x - \dfrac{x^2 - a}{2x}$ (Newton's method).

(i) For what values of a, if any, will the corresponding fixed point iterations be locally convergent to $\overline{x}$? Explain.
(ii) For what values of a, if any will the iterations in (i) be quadratically convergent.
(iii) Suppose $a = 5$. Show that $x_{k+1} = g_2(x_k)$ will converge for an arbitrary initial iterate $x_0 \in [2, 3]$. How many iterations will guarantee absolute error $< .01$?
(iv) Compute $\sqrt{5}$ to within absolute error 10^{-6} using one of the above g's.

IV.(i) Show that Newton's method as applied to

$$f(x) = \frac{1}{x} - a = 0, \qquad a \neq 0$$

leads to an algorithm for computing reciprocals $\frac{1}{a}$ without dividing.
(ii) Show that $r_{k+1} = r_k^2$, where r_k is the relative error in the iterate x_k. For what range of initial iterates x_0 will the iteration converge to $\frac{1}{a}$?
(iii) Suppose the algorithm in (i) is used to create a division capability for a binary computer which has a 53-bit mantissa and rounds. For $a = (1.d_1 d_2 \cdots d_{52})_2 \times 2^e$, the initial iterate is taken to be $x_0 = 2^{-(e+1)}$ (and analogously for $a < 0$). Bound the initial relative error r_0. How many iterations are needed to guarantee full machine precision?
(iv) Show, in general, that if $|e_{k+1}| \leq M e_k^2$ for all k, then $|e_k| \leq \frac{1}{M}(M e_0)^{2^k}$; thus $|e_k| \to 0$ if $|e_0| < \frac{1}{M}$.

V. We wish to determine the order of convergence of the secant method as applied to $f(x) = x^2 - a = 0$, $a > 0$.
(i) Show that $e_{k+1} = \frac{e_k \, e_{k-1}}{x_k + x_{k-1}}$. [Hint: First show $x_{k+1} = x_k - \frac{x_k^2 - a}{x_k + x_{k-1}}$, then subtract $\overline{x} = \sqrt{a}$ from both sides.]
(ii) What is the order of convergence of the iteration, i.e., the value of p for which $|e_{k+1}| = O(|e_k| \cdot |e_{k-1}|)$ is equivalent to $|e_k| = O(|e_{k-1}|^p)$? [Note: More generally, the secant method converges at this rate to simple roots ($f'(\overline{x}) \neq 0$).]

VI. Suppose $x_{k+1} = g(x_k)$ is converging linearly to $\overline{x}$, which we would like to compute to within absolute error ϵ. We ask: Is it safe to stop iterating when $|x_{k+1} - x_k| \leq \epsilon$, and then use x_{k+1} as the approximation for $\overline{x}$?

To answer this question, assume

$$(*) \qquad x_{k+1} - \overline{x} \cong g'(\overline{x})(x_k - \overline{x}), \quad 0 < |g'(\overline{x})| < 1,$$

which is valid for $x_k \cong \overline{x}$, and derive the relation

$$x_{k+1} - \overline{x} \cong \frac{-g'(\overline{x})}{1 - g'(\overline{x})}(x_{k+1} - x_k).$$

[Hint: Express $x_k - \overline{x}$ in $(*)$ in terms of $x_k - x_{k+1}$ and $\overline{x} - x_{k+1}$.]

For what values of $g'(\overline{x})$ in the range of interest is the above stopping condition is "safe"?