CS 323 Homework - due 2/2/12

SOLUTIONS

I. $\beta = 10$, $n = 4$, $m = -30$, $M = 30$, rounding ...

(a) $x = 1.6726231 \times 10^{-24}$

$\hat{x} = 1.673 \times 10^{-24}$, absolute error $= |\hat{x} - x| = 3.769 \times 10^{-28}$, relative error $= \left|\frac{\hat{x}-x}{x}\right| = 2.25 \times 10^{-4}$

(b) $x = 5.9742 \times 10^{27}$

$\hat{x} = 5.974 \times 10^{27}$, absolute error $= 2 \times 10^{23}$, relative error $= 3.35 \times 10^{-5}$

II. (i) $x = (23.625)_{10} = (10111.101)_2$; $y = (.8)_{10} = (.1100110011...)_2$.

(ii) $\beta = 2$, $n = 6$, rounding $\rightarrow \hat{x} = (10111.1)_2 = (23.5)_{10}$; $\hat{y} = (.110011)_2 = \frac{51}{64} = (.703125)_{10}$

(iii) $|\hat{x} - x| = .125$, $\left|\frac{\hat{x}-x}{x}\right| = 5.3 \times 10^{-3}$; $|\hat{y} - y| = .003125$, $\left|\frac{\hat{y}-y}{y}\right| = 3.9 \times 10^{-3}$

III. $\beta = 2$, $n = 3$, $m = -4$, $M = 4$, chopping ...

(i) computed value of $5 + 4 + 3 + 2 + 1$: 12

(ii) computed value of $1 + 2 + 3 + 4 + 5$: 14

(iii) computed value of $\overbrace{1 + \cdots + 1}^{\text{100 times}}$: 8

IV. $\mathcal{F}$: "0", $\{\pm(1.d_1 \ldots d_{52})_2 \times 2^e\}$, $d_i = 0$ or 1, $e \in [-1022, +1023]$.

(i) separation between 1 and next larger number in $\mathcal{F}$: 2^{-52}

(ii)

$$1 + 2^k = \begin{cases} (1.\overbrace{0\ldots0}^{\text{k-1}}1)_2 \times 2^k, & k \geq 1 \\ (1.\overbrace{0\ldots0}^{\text{k-1}}1)_2 \times 2^0, & k \leq -1 \end{cases}$$

$$\text{computed value of } 1 + 2^k : \begin{cases} 1 + 2^k, & k \in [-52, 52] \\ 2^k, & k > 52 \\ 1, & k < -52 \end{cases}$$

V. formulas for roots of $ax^2 + bx + c = 0$:

(1) $$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(2) $$x_\pm = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}.$$

$\beta = 10$, $n = 3$, rounding; $x^2 - 16x + 1 = 0$ ...

formula (1) gives: $x_+ = 16.0, x_- = .0500$
formula (2) gives: $x_+ = 20.0, x_- = .0627$
Correct values to 3 decimal digits: $x_+ = 15.9, x_- = .0627$
grossly inaccurate values caused by subtraction of almost equal numbers

VI.

$f(x) = \ln x \implies f'(x) = \frac{1}{x},\ f^{(2)}(x) = -\frac{1}{x^2},\ f^{(3)}(x) = \frac{2}{x^3}$

(i) Taylor polynomial for $f(x) = \ln x$ about $x = 1$: $p_2(x) = (x-1) - \frac{1}{2}(x-1)^2$.
$\ln 1.2 \approx p_2(1.2) = .18$

(ii) bound using error formula for Taylor polynomials:

$$|\ln 1.2 - p_2(1.2)| \leq (1.2-1)^3 \ \frac{\max_{\xi \in [1,1.2]} |f^3(\xi)|}{3!} = .008\,\frac{2}{6} = .00267$$

Check: correct value of $\ln 1.2$: .18232; actual absolute error in $p_2(1.2) = .18232 - .18 = .00232 < .00267$