

CS513: Homework 9 Solutions

1. • For any bin (except the last) produced by the NF-algorithm, the sum of the items in that bin and the first item in the next bin is > 1 . Adding this inequality over all bins, we see that on the left hand side, no item is counted more than twice in the worst case. Thus

$$2 \sum_{i=1}^n s_i > NF - 1$$

But $\sum_{i=1}^n s_i$ is a lower bound on OPT . Therefore,

$$NF - 1 < 2OPT \Rightarrow \frac{NF}{OPT} < 2$$

- Consider the $2n$ items with weights ordered as $(1/n, 1, 1/n, \dots, 1/n, 1)$. Here, $OPT = n + 1$ but $NF = 2n$. Therefore, as $n \rightarrow \infty$, the ratio $\frac{NF}{OPT}$ approaches 2.
 - One possible improvement to the NF algorithm would be to initially sort the input according to the weights. But I doubt this would change the approximation factor. Another algorithm is the First-fit algorithm, which considers items in an arbitrary order. In the i th step, it has a list of partially packed bins, say B_1, \dots, B_k . It attempts to put the next item, a_i , in one of these bins in this order. If a_i does not fit into any of these bins, it opens a new bin B_{k+1} , and puts a_i in it. This approach also gives a factor 2 approximation.
2. • Let there be n elements and m sets. We have two sets of variables, $x_i = 1$ iff set S_i is picked and $y_i = 1$ iff element e_i is covered. Then our IP is

$$\text{maximize } \sum_{i=1}^n y_i \tag{1}$$

$$\text{subject to: } \sum_{i=1}^m x_i = k \tag{2}$$

$$\forall 1 \leq i \leq n, \sum_{j|e_i \in S_j} x_j \geq y_i \tag{3}$$

$$\forall 1 \leq i \leq m, x_i \in \{0, 1\} \tag{4}$$

$$\forall 1 \leq i \leq n, y_i \in \{0, 1\} \tag{5}$$

Constraint (3) is because we want to pick an element only when it is covered by atleast one set that we have picked.

- We now relax the above IP to an LP by rewriting constraints (4) and (5) as $x_i \leq 1$ and $y_i \leq 1$. Let the solution of this LP be (x^*, y^*) . We will use randomized rounding by choosing set S_i with probability x_i^* . Let us call the collection of chosen sets as S^* and let S' be the set of covered elements due to S^* . Note that the expected number of sets we will choose will always be $\sum_{i=1}^m x_i^*$ which is equal to k .

Now consider an element e_i .

$$\Pr[e_i \in S'] = 1 - \Pr[e_i \notin S'] = 1 - \prod_{e_i \in S_j} (1 - x_j^*)$$

Using the fact that for any $0 \leq p \leq 1$, $1 - p \leq e^{-p}$ and constraint (3) we get

$$\Pr[e_i \in S'] \geq 1 - \prod_{e_i \in S_j} e^{-x_j^*} \geq 1 - e^{-\sum_{j|e_i \in S_j} x_j^*} \geq 1 - e^{-y_i^*}$$

Therefore, the expected number of elements covered

$$\mathbb{E}[|S'|] = \sum_{i=1}^n \Pr[e_i \in S'] \geq \sum_{i=1}^n 1 - e^{-y_i^*}$$

Now using the fact that for any $0 \leq q \leq 1$, $1 - e^{-q} \geq (1 - \frac{1}{e})q$ we get

$$\mathbb{E}[|S'|] \geq (1 - \frac{1}{e}) \sum_{i=1}^n y_i^* \geq (1 - \frac{1}{e})OPT$$

since OPT is a lower bound on the objective function. Hence we get a $(1 - \frac{1}{e})$ factor approximation.

3. We shall reduce this problem to minimum set cover. Our elements are the set S of rectangles. Each row or column can be reinterpreted as a subset of S , in particular, a subset corresponding to a row or column would be those rectangles that this row or column stabs. Then we use our $O(\log n)$ approximation for set cover. Minimizing $\max\{|R|, |C|\}$ is similar (upto a constant factor of 2) to minimizing $|R| + |C|$. Hence we get an $O(\log n)$ approximation.