CS 509: Take Home Mid-Term

Due October 17

Rules of the game: This is to be your work and your work only. You may not consult any other person regarding the questions in this exam. You may of course request clarification on these questions; all clarifications will be posted on the web site.

This exam is open book. However, you should not seek out specific solutions to these questions on the web, in textbook teacher's editions, etc.

Problem 1: For this problem, L_1 is an arbitrary context free language and L_2 is an arbitrary regular language. In class we learned that the intersection of a regular and a context free language is context free.

A. Show that it is decidable whether a regular language contains *all* strings of the form $a^n b^n$.

B. Show how to decide whether $L_1 \subseteq L_2$.

C. Show that it is undecidable whether $L_2 \subseteq L_1$.

Problem 2: Recalling Problem Set 2 (Sipser, 4.18): Let A and B be disjoint languages. We say that a language C *separates* A and B if $A \subseteq C$ and $B \subseteq \overline{C}$ ($\overline{C}$ denotes the complement of C).

A. Show that if A and B disjoint and either A or B is regular (or both), then there exists a regular language C that separates them.

B. Give with proof an example of disjoint A and B such that

- A and B are context free.
- Neither A nor B are regular.
- There exists a regular language C that separates A and B.

C. Give with proof an example of disjoint A and B such that

- A and B are context free.
- There is no regular C that separates A and B.

Problem 3: We consider the following two definition for a language L being decidable on a subset $S \subseteq \Sigma^*$:

Definition 1 *L is decidable on $S \subseteq \Sigma^*$ iff there exists a Turing machine M such that*

- *M halts on all inputs $x \in \Sigma^*$.*
- *For all $x \in S$, $x \in L$ iff M accepts on input x.*

Definition 2 *L is decidable on $S \subseteq \Sigma^*$ iff there exists a Turing machine M such that*

- *M halts on all inputs $x \in S$.*
- *For all $x \in S$, $x \in L$ iff M accepts on input x.*

When are these definitions equivalent?

A. Show that if S is decidable, then Definitions 1 and 2 are equivalent.

B. Give an example of a set S and language L that is decidable on S according to Definition 2 but not Definition 1.

Problem 4: Let $<_\ell$ denote lexicographic ordering on strings: $a <_\ell b$ if $|a| < |b|$ (shorter strings come first) and equal-length strings are compared according to alphabetical order. Let L be an infinite recursively enumerable language and let M enumerate L. That is, M writes on its output tape the infinite sequence $w_0, w_1, \ldots$, listing all the strings in L (every string in L is eventually output).

A. Show that if $w_0, w_1, \ldots$ are in lexicographic order ($w_i <_\ell w_{i+1}$), then L is decidable.

B. Show that there exists a Turing machine M' that outputs an infinite subset of L in lexicographic order. That is, it outputs an infinite sequence $q_0, q_1, \ldots$, such that $q_i <_\ell q_{i+1}$ and $q_i \in L$ for all i.

C. Conclude that every infinite recursively enumerable language has an infinite recursive subset. Note that this part is easy despite being the last part. You may assume parts A and B when answering this part.

Problem 5: Denote by $<M>$ the description of Turing machine M. Let $H(<M>) = 1$ if M halts (on no input) and $H(<M>) = 0$ otherwise. Define L by

$$L = \{(<M_1>, <M_2>) | H(<M_1>) \oplus H(<M_2>) = 1\}.$$

That is, L is the languages of pairs of (descriptions of) Turing machines such that exactly one of the Turing machines halts on no input.

A. Show that $\text{HALT}_0 \leq_m L$.

B. Show that $\overline{\text{HALT}_0} \leq_m L$.

C. Show that $L \in \Sigma_2$. Bonus: show that it is *not* Σ_2 complete.

Recall from class that HALT_0 is defined as

$$\text{HALT}_0 = \{<M> | M \text{ halts given an empty input}\}.$$

That is, $<M>$ is in HALT_0 iff when you start M with a blank input tape, it will eventually halt.