2004

Relations   sec. 7.1

Remember...

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$$

$A = \{1, 2, 3\}$

$B = \{x, y\}$

$A \times B = \{(1,x), (1,y), (2,x), (2,y), (3,x), (3,y)\}$

$B \times A = \{(x,1), (x,2), (x,3), (y,1), (y,2), (y,3)\}$

A binary relation from A to B is <u>a subset</u> of $A \times B$

$R_1 = \{(1,x), (2,y), (3,x)\}$

$R_2 = \{(1,y), (2,x)\}$

$R_3 = \{(2,y)\}$

$R_4 = \Phi = \{\}$

How to visualize relations?

Let $\mathcal{R}$ be a binary Relation from A to B

$$\mathcal{R} \subseteq A \times B$$

<u>Notation</u>

$a \, \mathcal{R} \, b$ $\qquad\qquad$ $(a, b) \in \mathcal{R}$

$a \, \not\mathcal{R} \, b$ $\qquad\qquad$ $(a, b) \notin \mathcal{R}$

Ex:

A : set of All cities

B : set of the 50 states

$\mathcal{R}$: $(a, b) \in \mathcal{R}$ if city a is in state b

(Red Bank, New Jersey) $\in \mathcal{R}$

(Ann Arbor, Michigan) $\in \mathcal{R}$

Boulder $\mathcal{R}$ Colorado

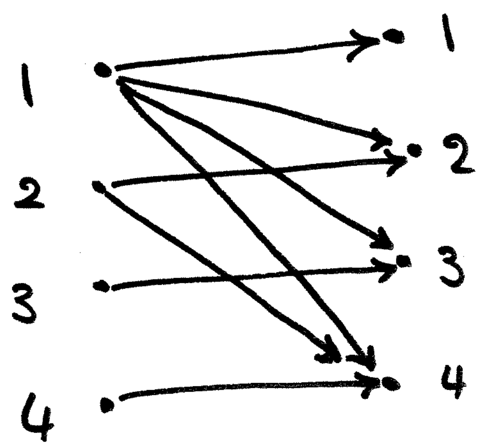
Relation on a set

$$R \subseteq A \times A$$

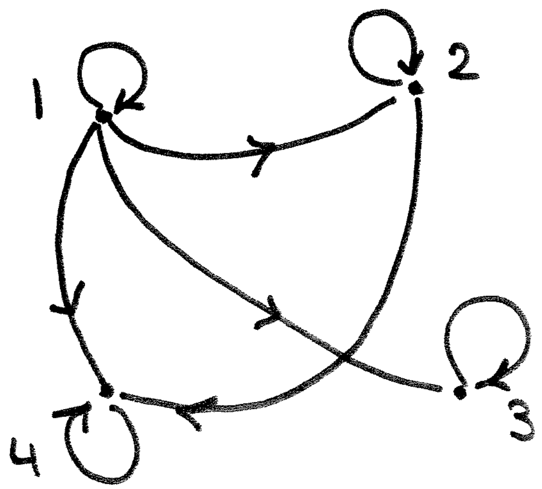
Ex : $A = \{1, 2, 3, 4\}$

aRb : a divides b

$$R = \{ (1,2), (1,3), (1,4), (1,1), (2,2), (2,4), \\ (3,3), (4,4) \}$$



R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x



How many relations on a set A with n elements

$$|A| = n$$

$$|A \times A| = n^2$$

any relation $R \subseteq A \times A$

How many subsets possible? 2^m

$\therefore$ number of possible relations $\underline{\underline{2^{n^2}}}$

A relation R on a set A is called

reflexive

if $(a,a) \in R$ for every element $a \in A$

Ex: let $A = \{1, 2, 3, 4\}$

$R_1 = \{\underline{(1,1)}, (1,2), (2,3), \underline{(2,2)}, (2,3), \underline{(3,3)}, \underline{(4,4)}\}$

← Reflexive

$R_2 = \{(1,2), (1,1), (2,3), (3,3), (2,4), (4,4)\}$

← not reflexive

Ex: "divides" relation on the set of positive integers

- A relation R on a set A is called **Symmetric** if $(b,a) \in R$ whenever $(a,b) \in R$ **for all** $a, b \in A$

- A relation R on a set A is called **Antisymmetric** if $(a,b) \in R$ and $(b,a) \in R$, then it must be that $a = b$ **for all** $a, b \in A$

(you can not find any (a,b) and (b,a) at the same time)

Symmetric & Antisymmetric are <u>not</u> Opposits.

We can have R that is Neither Symmetric Nor Anti Symmetric

R	1	2	3
1	x	x	x
2		x	
3	x		

$(1,3) \in R \quad (3,1) \in R$ NOT Anti Sym.

$(1,2) \in R \quad (2,1) \notin R$ Not Symm.

Can we have a relation that is both Symmetric & Anti Symm. ??

Yes ϕ

A relation R on a set A is called

<u>transitive</u>

if whenever $(a,b) \in R$ and $(b,c) \in R$

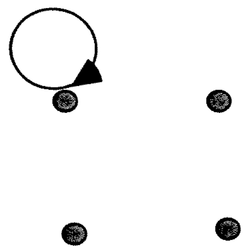
then $(a,c) \in R$ for all $a,b,c \in A$

Ex: divides

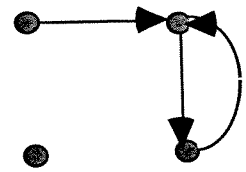
a divides b , b divides c then a divides c

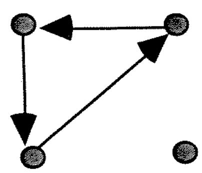
Examples:

A.

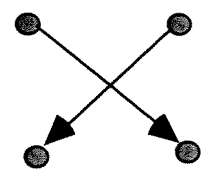


B.





C.



D.

A: not reflexive
symmetric
antisymmetric
transitive

B: not reflexive
not symmetric
not antisymmetric
not transitive

C: not reflexive
not symmetric
antisymmetric
not transitive

D: not reflexive
not symmetric
antisymmetric
transitive

$$A = \{1, 2, 3, 4\}$$

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$$

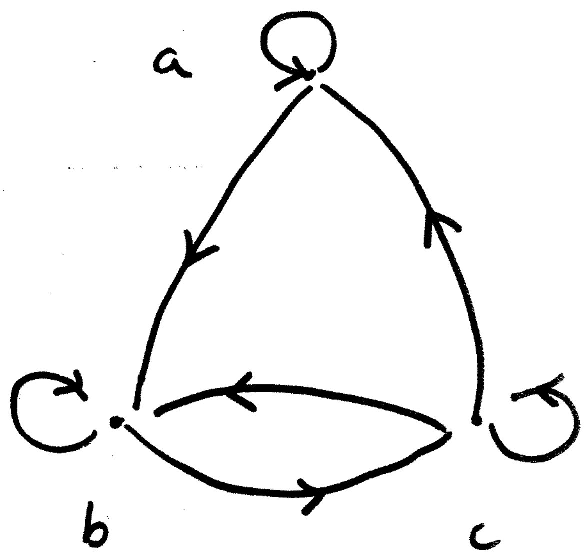
$$R_2 = \{(1,1), (1,2), (2,1)\}$$

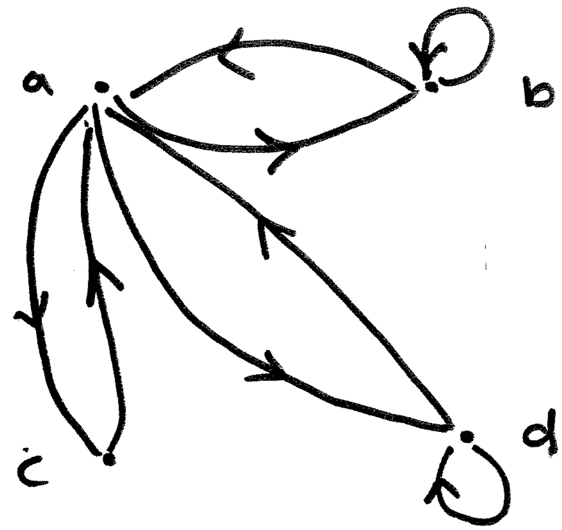
$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$$

$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$$

$$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

$$R_6 = \{(3,4)\}$$





ref. ✓

sym. ✗

Anti Sym. ✗

Trans. ✗

ref ✗

sym. ✓

Anti Sym. ✗

Trans ✗

Combining Relations:

since relations are sets we can use set operations on them.

$R_1 = \{(1,1), (2,2), (3,3)\}$

$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$

$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3),$
$(1,4)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)$

inverse relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

Complement Relation

$$\overline{R} = \{(a, b) \mid (a, b) \notin R\}$$

Composition

Definition: Suppose

- R1 is a relation from A to B
- R2 is a relation from B to C.

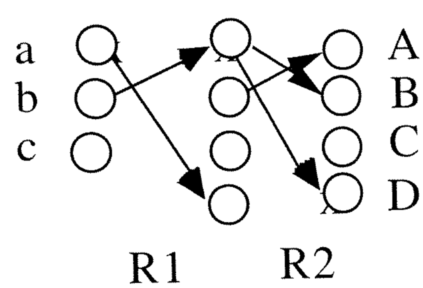
Then the composition of R2 with R1, denoted R2∘R1 is the relation from A to C:

If <x. y> is a member of R1 and <y, z> is a member of R2 then <x, z> is a member of R2∘R1.

Note: For <x, z> to be in the composite relation R2∘R1 there must exist a y in B . . . .

Note: We read them right to left as in functions.

Example:



R2∘R1 = {<b, D>, <b, B>}
