Homework due Wednesday, December 2

Problems 1-3 will be graded by a different person than problems 4-5. Thus please hand these in on separate pieces of paper.

1. Miscellaneous Exercise 101(a).

2. We’ve seen examples of co-r.e. sets that have no infinite r.e. subset. In contrast, show that every infinite r.e. set has an infinite decidable subset. (Hint: consider problem 5 on the September 14 homework.)

3. Let \( R_C = \{ x : C(x) \geq |x| \} \) be the set of Kolmogorov-random strings. We’ve seen that \( R_C \) has no infinite r.e. subset. Thus \( R_C \) is not r.e. But we did not show that you can reduce (the complement of) the Halting Problem to \( R_C \). There is a reason for this: The complement of the Halting Problem is not \( \leq_m \)-reducible to \( R_C \). Prove the following two (slightly easier) statements:
   3.a. \( R_C \) is co-r.e.
   3.b. There is not any one-one computable function \( f \) such that \( x \) is not in the Halting Problem iff \( f(x) \in R_C \).

Remember: These last two problems should be handed in separately from the first two.

4. Consider the following “approximations” to the set of K-random strings:
   Let \( R^m \) be \( \{ x : C^m(x) \geq |x| \} \), where \( C^m(x) = \min \{ |d| : U(d) = x \text{ in at most } m \text{ steps} \} \).
   Show that, if \( x \) is the lexicographically-first string of length \( n \) in \( R^m \), then \( C(x) \leq C(m) + 2 \log n + O(1) \).

5. Show that the following oracle Turing machine, with an oracle for the set of K-random strings, is a Turing reduction of the Halting Problem to \( R_C \):
   1. On input \((x, y)\), we want to use the oracle to help us decide of \( M_x \) halts on input \( y \).
   2. Let \(|(x, y)| = n|.
   3. For all strings \( z \) of length \( 5n \) such that \( z \notin R_C \) search for a description \( d \) of length less than \( 5n \) such that \( U(d) = z \). Let \( n_z \) be the number of steps that \( U \) makes in the computation \( U(d) = z \).
   4. Let \( m = \max_z n_z \).
   5. Run \( M_x(y) \) for \( m \) steps. If the computation halts in this time, then output “YES”. Otherwise, output “NO”.

Hint: Make use of problem 4.