Homework due Wednesday, September 23

**Motivation, Definitions and Notation:** Recall that the proof of Gödel’s Incompleteness Theorem involved the construction of a formula \( \phi_{M,x} \) of the form \( \phi_{M,x} = \neg \exists v \text{ VALCOMP}_{M,x}(v) \) with the property that \( \mathbb{N} \models \phi_{M,x} \) if and only if Turing machine \( M \) does not halt on input \( x \). (Here, I’m blending and slightly altering the notation that I used in lecture, to make things align more closely to the proof presented by Kozen.) Let PA denote Peano Arithmetic. The Incompleteness Theorem says that there is a Turing machine \( M \) and an input \( x \) such that \( \mathbb{N} \models \phi_{M,x} \), but such that there is no proof in PA of the logic formula \( \phi_{M,x} \). That is, the machine \( M \) really does not halt on input \( x \), but this cannot be proved in PA.

For the remainder of this write-up, we will use the notation “\( M \)” and “\( x \)” to refer to this specific machine \( M \), which does not halt on input \( x \), but where this cannot be proved in PA.

Let \( \Gamma \) be any set of formulae. Recall that the notation \( \Gamma \models \phi \) means that, for every structure \( M \) such that \( M \models \Gamma \), it also holds that \( M \models \phi \). In particular, if there is no structure that satisfies \( \Gamma \), then the condition “\( M \models \phi \)” is considered to hold vacuously (since it holds for “every structure that satisfies \( \Gamma \)”), and thus, in this case we say that \( \Gamma \models \phi \) for every formula \( \phi \), including obvious contradictions such as the case when \( \phi \) is \( \psi \land \neg \psi \). In this case, we say that \( \Gamma \) is **inconsistent**.

The notation \( \Gamma \vdash \phi \) means that there is a proof (using some standard notion of “proof system”) such that every line of the proof is either an element of \( \Gamma \) or else follows from some earlier lines according to the inference rules of the proof system, where the final line in the proof is \( \phi \).

**Gödel’s Completeness Theorem** says that \( \Gamma \models \phi \) implies \( \Gamma \vdash \phi \).

In particular, the proof of the Completeness Theorem establishes that, if \( \Gamma \) is inconsistent, then \( \Gamma \vdash (\psi \land \neg \psi) \). This proof must be of finite length, and thus it can only make use of finitely-many of the formulae in \( \Gamma \). This proves:

**The Compactness Theorem for first-order logic:** If \( \Gamma \) is inconsistent, then there must be a finite subset \( \Gamma' \subseteq \Gamma \) that is inconsistent (because \( \Gamma' \vdash (\psi \land \neg \psi) \)).

For the particular case where \( \Gamma = \text{PA} \), combining the completeness theorem and the incompleteness theorem, we have that \( \text{PA} \not\vdash \phi_{M,x} \), and thus PA \( \not\models \phi_{M,x} \) – and thus (by definition) there must be a structure \( M \) such that \( M \models \text{PA} \) and simultaneously \( M \models \neg \phi_{M,x} \). That is, \( M \) “looks like” \( \mathbb{N} \) (in the sense that it satisfies PA, which are the usual axioms for \( \mathbb{N} \)), but nonetheless \( M \models \exists v \text{ VALCOMP}_{M,x}(v) \). That is, in the structure \( M \), there is a “number” that encodes a halting computation transcript of \( M \) on input \( x \). This homework assignment asks you to explore how this can possibly be.
Problems 1 and 2 will be graded by a different person than problems 3 and 4. Thus please hand these in on separate pieces of paper.

1. Let $\Gamma_0$ be PA. Let $c$ be a new constant. Consider $\Gamma_1 = \Gamma_0 \cup \{c > 1, c > 1 + 1, c > 1 + 1 + 1, \ldots \}$. Prove that every finite subset of $\Gamma_1$ is consistent. Then explain why this shows that there is a structure $\mathcal{M}$ such that $\mathcal{M} \models \Gamma_1$.

$\mathcal{M}$ is called a non-standard model of arithmetic. Note that $\mathcal{M}$ contains elements that are larger than any “standard” integer.

In the rest of this assignment, fix one such “non-standard” structure $\mathcal{M}$.

2. PA proves several standard facts about $\mathbb{N}$, such as

$$\forall x \ (x > 0 \rightarrow (\exists y \ y + 1 = x \wedge y \neq x))$$

and $\forall x \forall y \ (x < y \lor y < x \lor y = x)$, and $\forall x \exists y \ (y^2 \leq x \wedge (y + 1)^2 > x)$.

(This third statement can be interpreted as saying “for all $x$, $\lceil \sqrt{x} \rceil$ exists”.) What does this allow you to say about the interpretation of the constant $c$ in $\mathcal{M}$? Do the elements $c - 1$ and $\lceil (c)^{1/2} \rceil$ exist? If so, are these elements standard or non-standard elements? Is it correct to think of $\mathcal{M}$ as being $\mathbb{N} \cup \{\infty\}$, or is it something more complicated?

Remember: These last two problems should be handed in separately from the first two.

3. Consider the closed terms that do not use the constant $c$ (such as $(1 + 1) \times (1 + 1 + 1)$). Do these terms represent standard or non-standard elements of $\mathcal{M}$?

4. Now consider any structure $\mathcal{M}'$ such that $\mathcal{M}' \models \exists v \ \text{VALCOMP}_{M,x}(v)$ and $\mathcal{M}' \models \text{PA}$. (That is, we are not necessarily assuming ahead of time that $\mathcal{M}'$ satisfies $\Gamma_1$ or even that $\mathcal{M}'$ has an interpretation of the constant $c$.) Show that we can nonetheless conclude that $\mathcal{M}'$ contains nonstandard elements. (Hint: consider the formulas of the form $\text{VALCOMP}_{M,x}(1 + 1 + 1 + 1)$, and with other closed terms plugged in place of the variable $v$.)