The Recursion Theorem and Gödel’s Incompleteness Theorem

In these notes, we show how to use the Recursion Theorem to prove the stronger version of Gödel’s Incompleteness Theorem.

Let $M_1$ be the Turing machine that takes as input the pair $(M, j)$ and performs the following operations:

- Let $\psi = \neg \exists v \text{VALCOMP}_{M,j}(v)$.

- Search for a proof in PA of $\psi$. If such a proof is found, halt and output “1”.

Remark: Note that $M_1$ halts on input $(M, j)$ if and only if there is a proof in Peano Arithmetic of the statement that means “$M$ does not halt on input $j$”.

Now let $\sigma$ be the function that takes $i$ as input, and outputs the code for $M_1$, with “$i$” hardwired in for the variable $M$, and with “$i$” also hardwired in for the variable $j$. That is, $\sigma$ takes $i$ as input produces as output a program (i.e., an index for a Turing machine) that performs the following operations:

- Let $\psi = \neg \exists v \text{VALCOMP}_{i,i}(v)$.

- Search for a proof in PA of $\psi$. If such a proof is found, halt and output “1”.

Note that this program does not look at its input. That is, either $M_\sigma(i)$ halts for every input $x$, or it runs forever for every input $x$.

Now let $i$ be an index (guaranteed to exist, by the Recursion Theorem) such that $\phi_i = \phi_{\sigma(i)}$. Is $\phi_i$ defined?

Note that $\phi_i(x)$ is defined only if $M_i$ halts on input $x$, which happens if and only if $M_{\sigma(i)}$ halts (and this does not depend on $x$). Looking at the program $\sigma(i)$, $M_{\sigma(i)}$ halts if and only if there is a proof in PA of the statement $\neg \exists v \text{VALCOMP}_{i,i}(v)$. If PA is consistent, this happens only if machine $M_i$ does not halt on input $i$. That is, we have concluded that if $\phi_i(x)$ is defined for some $x$, then $\phi_i(x)$ is defined for every $x$, which in turn implies that $\phi_i(i)$ is not defined (if PA is consistent). This is a contradiction (if PA is consistent).

Thus we must conclude that $M_i$ does not halt for any $x$, including $x = i$. Thus $\neg \exists v \text{VALCOMP}_{i,i}(v)$ is true. But since $M_i$ and $M_{\sigma(i)}$ compute the same partial function, this means that $M_{\sigma(i)}$ does not halt, which (looking at the code for $M_{\sigma(i)}$) means that there is not a proof in PA of the sentence $\neg \exists v \text{VALCOMP}_{i,i}(v)$.

That is, assuming PA is consistent, $\neg \exists v \text{VALCOMP}_{i,i}(v)$ is a true statement that is not provable in PA. And since the proof of the Recursion Theorem allows us to construct this value “$i$”, we have a completely explicit example of such a sentence.