

## LECTURE NOTES

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<p><b>Theorem</b> For <math>k = n^{\frac{1}{4}}</math>, <math>k</math>-clique requires monotone circuits of size <math>n^{\Omega(\sqrt{k})}</math></p>
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**Remark** If this could be done for *every* monotone problem in  $NP$ , this would imply  $P \neq NP$ .

A *slice function*  $f$  is one function such that for all  $n$ , there exists  $m$  such that

$$\begin{aligned} f(x) &= 0 & \text{if } \sum x_i < m \\ f(x) &= 1 & \text{if } \sum x_i > m \end{aligned}$$

Note that all slice functions are monotone. There are slice functions that are NP-complete. One can prove that the monotone circuit complexity of a slice function  $f$  is not much greater than the (non-monotone) circuit complexity of  $f$ . Thus we may expect that it is not always easy to prove lower bounds on monotone circuit complexity.

**Basic idea:** Every monotone circuit for  $k$ -clique  $C$  is “approximated” by a monotone depth 2 circuit  $\tilde{C}$ . The circuit has  $m$  “ $\wedge$ ” gates with fan in  $\leq \binom{l}{2}$  on the first level, each detects a clique of size no more than  $l (= \sqrt{k})$ . All the outputs of the  $m$  “ $\wedge$ ” gates connected to an “ $\vee$ ” gate. (The circuit just check the presence, but not absence).

What we will show is

1. If  $C$  is a small monotone circuit, then for most  $k$ -cliques  $G$ ,  $\tilde{C}(G) \leq C(G)$  and for most complete  $(k-1)$ -partite graphs  $G$ ,  $\tilde{C}(G) \geq C(G)$
2. Every approximator either outputs 0 on most  $k$ -cliques or outputs 1 on most  $(k-1)$ -partite graphs

First we are going to prove the following lemma:

**Sunflower Lemma** Let  $F$  be a collection of sets, each of size  $\leq l$ . If  $|F| > (p-1)^l l!$ , then there is a sunflower with  $p$  petals. (A sunflower with  $p$  petals is a collection of sets  $S_1, \dots, S_p$  such that for all  $i < j$  and  $k < l$ ,  $S_i \cap S_j = S_k \cap S_l$ .)

**Proof** By induction on  $l$ :

**Basis:**  $l = 1$ , choose  $p$  disjoint singletons.

**Induction:** Let  $M$  be a maximal collection of disjoint sets in  $F$ ,  $M = \{A_1, A_2, \dots, A_r\}$ , Let  $S = \bigcup_{i=1}^r A_i$ . If  $r \geq p$  we have done.

Otherwise  $|S| \leq l(p-1)$ . Since  $M$  is maximal,  $S$  intersects every element of  $F$ . There is some  $i \in S$ , such that  $i$  is in  $\geq 1/(p-1)l$  of the elements of  $F$ .

Let  $F' = \{B - \{i\} : i \in B \in F\}$ .  $|F'| \geq |F|/(p-1)l > \frac{(p-1)^l l!}{(p-1)} = (p-1)^{(l-1)}(l-1)!$

Let  $B_1 - \{i\}, B_2 - \{i\}, \dots, B_p - \{i\}$  be a sunflower in  $F'$ . Then  $B_1, B_2, \dots, B_p$  is our sunflower.

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We will build  $\tilde{C}$  from  $C$ , Starting at the leaves, that is, for each gate  $g$  of  $C$ , consider the function computed at  $g$ , we will build an approximator for that function.

- If  $g$  is a leaf, we are done.
- If  $g$  is an  $\vee$  gate,  $g = h_1 \vee h_2$ , define  $g = h_1 \sqcup h_2$  as follows:

1. If  $h_1 = \{S_1, S_2, \dots, S_{m_1}\}$ ,  $h_2 = \{T_1, T_2, \dots, T_{m_2}\}$  ( $S_i, T_j$ ,  $1 \leq i \leq m_1$ ,  $1 \leq j \leq m_2$  are sets of vertices (each set of size at most  $l$ ) and  $|h_1 \cup h_2| \leq m$ ), then set  $g = h_1 \cup h_2$ . Otherwise  $h_1 \cup h_2$  contains a sunflower of  $p$  petals. Replace those  $p$  sets with their center.
  2. If there are still more than  $m$  sets, repeat it until there are no more sunflowers.
- If  $g = h_1 \wedge h_2$ , define  $h_1 \sqcap h_2$  as follows:  
 Let  $h_1 = \{S_1, S_2, \dots, S_{m_1}\}$ ,  $h_2 = \{T_1, T_2, \dots, T_{m_2}\}$ ,  $h_1 \wedge h_2 = \{S_i \wedge T_j : 1 \leq i \leq m_1, 1 \leq j \leq m_2\}$ 
    1. Consider  $\{S_i \cup T_j : 1 \leq i \leq m_1, 1 \leq j \leq m_2\}$
    2. remove any such  $S_i \cup T_j$ , such that  $|S_i \cup T_j| > l$ ,
    3. replace sunflowers with their centers(plucking), until there are no more sunflowers (so there are  $\leq m$  sets remaining).

In what follows, we will use the following values for  $l, p$ , and  $m$ : ( $l = \sqrt{k}, p = 10\sqrt{k} \log n, m = (p - 1)^{l!}$ )

**Lemma 1** Every  $\tilde{C}$  is either identically 1, or it outputs 1 on at least half of the  $(k - 1)$ -coloured graphs

**Lemma 2** For all but at most

$$Size(C) \cdot m^2 \binom{n-l-1}{k-l-1} \leq Size(C) \cdot m^2 \left(\frac{k}{n}\right)^{l+1} \binom{n}{k}$$

$k$ -cliques  $G$ ,  $C(G) \leq \tilde{C}(G)$

**Lemma 3** For all but at most

$$Size(C) \cdot \left\lceil \frac{m^2}{2^p} \right\rceil (k-1)^n$$

$(k-1)$ -partite graphs  $G$ ,  $C(G) \geq \tilde{C}(G)$

**Theorem** For  $k \leq n^{\frac{1}{4}}$ , monotone circuits for  $k$ -clique need size  $n^{\Omega(\sqrt{k})}$

Before proving these lemmas, we will first see how they lead us to our theorem.

**Proof** By Lemma 1, we have 2 cases, if  $\tilde{C}$  is identically 1, then Lemma 2 says that

$$Size(C) m^2 \left(\frac{k}{n}\right)^{l+1} \binom{n}{k} \geq \binom{n}{k}.$$

(That is, the set of cliques such that  $C(g) > \tilde{C} = 0$  is the set of *all* cliques.)  
So

$$Size(C) \geq \frac{\left(\frac{n}{k}\right)^{l+1}}{m^2} \geq \frac{(n^{\frac{3}{4}})^{\sqrt{k}}}{[(10\sqrt{k} \log n - 1)^{\sqrt{k}} \cdot \sqrt{k}!]^2} \geq \frac{n^{\frac{3}{4}\sqrt{k} - \frac{5}{8}\sqrt{k}}}{100} = n^{\Omega(\sqrt{k})}$$

If case 2 holds, then by lemma 3,

$$Size(C) \cdot \left\lceil \frac{m^2}{2^p} \right\rceil (k-1)^n \geq \frac{1}{2} (k-1)^n$$

$$Size(C) \geq \frac{1}{2} \cdot \frac{2^p}{m^2} \geq \frac{1}{2} \cdot \frac{2^{(10\sqrt{k} \log n)}}{[(10\sqrt{k} \log n - 1)^{\sqrt{k}} \cdot \sqrt{k}!]^2} \geq \frac{1}{2} \frac{n^{10\sqrt{k} - \frac{5}{8}\sqrt{k}}}{100} = n^{\Omega(\sqrt{k})}$$

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Now we are going to prove the lemmas:

**Proof of Lemma 1**

$$\tilde{C} = \bigvee_{i=1}^r (\text{clique on set } A_i), |A_i| \leq l, 1 \leq i \leq r$$

If  $r = 0$ , then claim holds.

$r = 1$ , Let  $A_1$  be given, then  $C_1$  accepts  $G$  iff there is a clique on  $A_1$  iff all vertices in  $A_1$  have different colours.

Consider a randomly chosen  $(k-1)$ -coloured graph, Probability that  $G$  is not accepted

$$\begin{aligned} &= Prob[\exists i, j \in A_1, i, j \text{ have the same colour}] \\ &\leq \binom{l}{2} \frac{1}{k-1} \\ &= \frac{\sqrt{k}(\sqrt{k}-1)}{2} \cdot \frac{1}{k-1} \\ &= \frac{k - \sqrt{k}}{k-1} \cdot \frac{1}{2} < \frac{1}{2} \end{aligned}$$

If  $r > 1$ , the result also holds, because  $\tilde{C}$  accepts at least as many  $G$  as in the case above. 1

**Proof of Lemma 2** If  $G$  is the  $k$ -clique, such that

$$C(G) = 1 \text{ but } \tilde{C}(G) = 0$$

then it must be the case that there is some gate  $g$  in  $C$  such that  $g = h_1 \vee h_2$  and in  $\tilde{C}$  the approximator built for  $g$  rejects  $G$  but the approximator for

one of  $h_1, h_2$  accepts  $G$ , or  $g = h_1 \wedge h_2$  and both  $h_1$  and  $h_2$  accept  $G$  but  $g$  rejects.

We will show that any such  $g$  can affect no more than

$$m^2 \left(\frac{k}{n}\right)^{l+1} \binom{n}{k}$$

cliques.

Note that if our approximator on  $h_1$  accepts  $G$ , that means that there is some clique  $A$  included in the clique in  $G$ , that is detected by our approximator for  $h_1$ . But our approximator for  $h_1 \vee h_2$  looks for a clique on some subset of  $A$ .

So in case 1, our approximator for  $g \geq h_1 \sqcup h_2$ . So the first problem gate is not of type 1.

Now consider  $g = h_1 \wedge h_2$ , Let  $A$  be the set of vertices containing the  $k$ -clique  $G$ . We are assuming that there is some  $S_i \subseteq A$  and some  $T_j \subseteq A$ , so that  $\tilde{h}_1$  detects a clique on  $S_i$  and  $\tilde{h}_2$  detects a clique on  $T_j$ . But our approximator  $\tilde{g}$  rejects  $A$ . Note that in  $G$ , there is a clique on  $S_i \cup T_j$ . It must be the case that  $|S_i \cup T_j| > l$ , because otherwise  $\tilde{g}$  accepts  $G$ . Note that there are at most  $m^2$  such  $(s_i, t_j)$  pairs associated with gate  $g$ .

There are no more than  $\binom{n-l-1}{k-l-1}$  such  $k$ -cliques in  $G$  for fixed  $S_i$  and  $T_j$ . Thus the number of such  $k$ -cliques is no more than

$$m^2 \binom{n-l-1}{k-l-1} \leq m^2 \left(\frac{k}{n}\right)^{l+1} \binom{n}{k}$$

□

**Proof of Lemma 3** If  $G$  is a  $(k-1)$ -coloured graph such that

$$C(G) = 0 \text{ but } \tilde{C}(G) = 1$$

then it must be the case as above

1.  $g = h_1 \vee h_2$  such that  $\tilde{g}$  accepts  $G$  but  $\tilde{h}_1$  and  $\tilde{h}_2$  reject  $G$ .
2.  $g = h_1 \wedge h_2$  such that  $\tilde{g}$  accepts  $G$  but one of  $\tilde{h}_1$  or  $\tilde{h}_2$  reject  $G$ .

We will show that any such  $g$  can affect no more than

$$\frac{m^2}{2^p}(k-1)^n$$

of the  $(k-1)$ -coloured graphs.

First consider  $g = h_1 \vee h_2$ . So  $\tilde{g}$  is built from  $\tilde{h}_1$  and  $\tilde{h}_2$  from plucking.

Let

$$\tilde{h}_1 = \bigvee(\text{clique on } S_1, S_2, \dots, S_{m_1})$$

$$\tilde{h}_2 = \bigvee(\text{clique on } T_1, T_2, \dots, T_{m_2})$$

$\tilde{h}_1$  and  $\tilde{h}_2$  each reject  $G$  iff each  $S_i$  and  $T_j$  are *not* “properly coloured”.

This means that there is some sunflower in the list  $S_1, S_2, \dots, S_{m_1}$  and  $T_1, T_2, \dots, T_{m_2}$  such that  $G$  properly colours the center but  $G$  does not properly colour any of the petals.

Let  $A_1, A_2, \dots, A_p$  be the sets in this sunflower.

$$\begin{aligned} & \text{Prob}[\text{none of } A_1, A_2, \dots, A_p \text{ is properly coloured but the centre is}] \\ & \leq \text{Prob}[A_1, A_2, \dots, A_p \text{ is not properly coloured} | \text{the centre is properly coloured}] \\ & = \prod_{i=1}^n \text{Prob}[A_i \text{ is not properly coloured} | \text{the centre is properly coloured}] \\ & \leq \prod_{i=1}^n \text{Prob}[A_i \text{ is not properly coloured}] \leq \frac{1}{2^p} \end{aligned}$$

And at last, if we notice that we will do plucking for  $m^2$  time, the result follows naturally.  $\square$