Algorithmic information, plane Kakeya sets, and conditional
dimension

Neil Lutz
Rutgers University

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Abstract

We formulate the conditional Kolmogorov complexity of $x$ given $y$ at precision $r$, where $x$ and $y$ are points in Euclidean spaces and $r$ is a natural number. We demonstrate the utility of this notion in two ways.

1. We prove a point-to-set principle that enables one to use the (relativized, constructive) dimension of a single point in a set $E$ in a Euclidean space to establish a lower bound on the (classical) Hausdorff dimension of $E$. We then use this principle, together with conditional Kolmogorov complexity in Euclidean spaces, to give a new proof of the known, two-dimensional case of the Kakeya conjecture. This theorem of geometric measure theory, proved by Davies in 1971, says that every plane set containing a unit line segment in every direction has Hausdorff dimension 2.

2. We use conditional Kolmogorov complexity in Euclidean spaces to develop the lower and upper conditional dimensions $\dim(xy)$ and $\Dim(xy)$ of $x$ given $y$, where $x$ and $y$ are points in Euclidean spaces. Intuitively these are the lower and upper asymptotic algorithmic information densities of $x$ conditioned on the information in $y$. We prove that these conditional dimensions are robust and that they have the correct information-theoretic relationships with the well studied dimensions $\dim(x)$ and $\Dim(x)$ and mutual dimensions $\mdim(x : y)$ and $\Mdim(x : y)$.

Joint work with Jack Lutz.